2 LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

1. (a) Using \( P(15, 250) \), we construct the following table:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( Q )</th>
<th>slope ( = m_{PQ} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(5, 694)</td>
<td>( \frac{694 - 250}{5 - 15} = -44.4 )</td>
</tr>
<tr>
<td>10</td>
<td>(10, 444)</td>
<td>( \frac{444 - 250}{10 - 15} = -38.8 )</td>
</tr>
<tr>
<td>20</td>
<td>(20, 111)</td>
<td>( \frac{111 - 250}{20 - 15} = -27.8 )</td>
</tr>
<tr>
<td>25</td>
<td>(25, 28)</td>
<td>( \frac{28 - 250}{25 - 15} = -22.2 )</td>
</tr>
<tr>
<td>30</td>
<td>(30, 0)</td>
<td>( \frac{0 - 250}{30 - 15} = -16.7 )</td>
</tr>
</tbody>
</table>

(c) From the graph, we can estimate the slope of the tangent line at \( P \) to be \( \frac{-300}{9} = -33.3 \).

2. (a) The slope appears to be \( \frac{1}{4} \).

(b) Using the values of \( t \) that correspond to the points closest to \( P \) \((t = 10 \) and \( t = 20) \), we have

\[
\frac{-38.8 + (-27.8)}{2} = -33.3
\]

(c) \( y = \frac{1}{4} (x - 1) \text{ or } y = \frac{1}{4}x + \frac{1}{4} \).

3. (a)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( Q )</th>
<th>( m_{PQ} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) 0.5</td>
<td>(0.5, 0.33333)</td>
<td>0.33333</td>
</tr>
<tr>
<td>(ii) 0.9</td>
<td>(0.9, 0.473684)</td>
<td>0.263158</td>
</tr>
<tr>
<td>(iii) 0.99</td>
<td>(0.99, 0.497487)</td>
<td>0.251256</td>
</tr>
<tr>
<td>(iv) 0.999</td>
<td>(0.999, 0.499750)</td>
<td>0.250125</td>
</tr>
<tr>
<td>(v) 1.5</td>
<td>(1.5, 0.6)</td>
<td>0.2</td>
</tr>
<tr>
<td>(vi) 1.1</td>
<td>(1.1, 0.523810)</td>
<td>0.238095</td>
</tr>
<tr>
<td>(vii) 1.01</td>
<td>(1.01, 0.502488)</td>
<td>0.248756</td>
</tr>
<tr>
<td>(viii) 1.001</td>
<td>(1.001, 0.500250)</td>
<td>0.249875</td>
</tr>
</tbody>
</table>

5. (a) \( y = y(t) = 40t - 16t^2 \). At \( t = 2 \), \( y = 40(2) - 16(2)^2 = 16 \). The average velocity between times 2 and \( 2 + h \) is

\[
v_{\text{ave}} = \frac{y(2 + h) - y(2)}{(2 + h) - 2} = \frac{40(2 + h) - 16(2 + h)^2 - 16}{h} = -24h - 16h^2 \text{ if } h \neq 0.
\]

(i) \([2, 2.5]: h = 0.5, v_{\text{ave}} = -32 \text{ ft/s}\)

(ii) \([2, 2.1]: h = 0.1, v_{\text{ave}} = -25.6 \text{ ft/s}\)

(iii) \([2, 2.05]: h = 0.05, v_{\text{ave}} = -24.8 \text{ ft/s}\)

(iv) \([2, 2.01]: h = 0.01, v_{\text{ave}} = -24.16 \text{ ft/s}\)

(b) The instantaneous velocity when \( t = 2 \) \((h \text{ approaches } 0)\) is \(-24 \text{ ft/s}\).
CHAPTER 2 LIMITS AND DERIVATIVES

7. (a) (i) On the interval \([1, 3]\), \(v_{\text{ave}} = \frac{s(3) - s(1)}{3 - 1} = \frac{10.7 - 1.4}{2} = \frac{9.3}{2} = 4.65 \text{ m/s.}\)

(ii) On the interval \([2, 3]\), \(v_{\text{ave}} = \frac{s(3) - s(2)}{3 - 2} = \frac{10.7 - 5.1}{1} = 5.6 \text{ m/s.}\)

(iii) On the interval \([3, 5]\), \(v_{\text{ave}} = \frac{s(5) - s(3)}{5 - 3} = \frac{25.8 - 10.7}{2} = \frac{15.1}{2} = 7.55 \text{ m/s.}\)

(iv) On the interval \([3, 4]\), \(v_{\text{ave}} = \frac{s(4) - s(3)}{4 - 3} = \frac{17.7 - 10.7}{1} = 7 \text{ m/s.}\)

(b) Using the points \((2, 4)\) and \((5, 23)\) from the approximate tangent line, the instantaneous velocity at \(t = 3\) is about \(\frac{23 - 4}{5 - 2} \approx 6.3 \text{ m/s.}\)

9. (a) For the curve \(y = \sin(10\pi/x)\) and the point \(P(1, 0)\):

<table>
<thead>
<tr>
<th>(x)</th>
<th>(Q)</th>
<th>(m_{PQ})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(2, 0)</td>
<td>0</td>
</tr>
<tr>
<td>1.5</td>
<td>(1.5, 0.8660)</td>
<td>1.7321</td>
</tr>
<tr>
<td>1.4</td>
<td>(1.4, -0.4339)</td>
<td>-1.0847</td>
</tr>
<tr>
<td>1.3</td>
<td>(1.3, -0.8230)</td>
<td>-2.7433</td>
</tr>
<tr>
<td>1.2</td>
<td>(1.2, 0.8660)</td>
<td>4.3301</td>
</tr>
<tr>
<td>1.1</td>
<td>(1.1, -0.2817)</td>
<td>-2.8173</td>
</tr>
</tbody>
</table>

As \(x\) approaches 1, the slopes do not appear to be approaching any particular value.

(b) We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at \(P\) that we need to take \(x\)-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose \(x = 1.001\), then the point \(Q\) is \((1.001, -0.0314)\) and \(m_{PQ} \approx -31.3794\). If \(x = 0.999\), then \(Q\) is \((0.999, 0.0314)\) and \(m_{PQ} = -31.4422\). The average of these slopes is \(-31.4108\). So we estimate that the slope of the tangent line at \(P\) is about \(-31.4\).
2.2 The Limit of a Function

1. As \( x \) approaches 2, \( f(x) \) approaches 5. [Or, the values of \( f(x) \) can be made as close to 5 as we like by taking \( x \) sufficiently close to 2 (but \( x \neq 2 \).] Yes, the graph could have a hole at (2, 5) and be defined such that \( f(2) = 3 \).

3. (a) \( \lim_{x \to -3} f(x) = \infty \) means that the values of \( f(x) \) can be made arbitrarily large (as large as we please) by taking \( x \) sufficiently close to -3 (but not equal to -3).

(b) \( \lim_{x \to 4^+} f(x) = -\infty \) means that the values of \( f(x) \) can be made arbitrarily large negative by taking \( x \) sufficiently close to 4 through values larger than 4.

5. (a) \( f(x) \) approaches 2 as \( x \) approaches 1 from the left, so \( \lim_{x \to 1^-} f(x) = 2 \).

(b) \( f(x) \) approaches 3 as \( x \) approaches 1 from the right, so \( \lim_{x \to 1^+} f(x) = 3 \).

(c) \( \lim_{x \to 1} f(x) \) does not exist because the limits in part (a) and part (b) are not equal.

(d) \( f(x) \) approaches 4 as \( x \) approaches 5 from the left and from the right, so \( \lim_{x \to 5} f(x) = 4 \).

(e) \( f(5) \) is not defined, so it doesn’t exist.

7. (a) \( \lim_{t \to 0^-} g(t) = -1 \) (b) \( \lim_{t \to 0^+} g(t) = -2 \)

(c) \( \lim_{t \to 0} g(t) \) does not exist because the limits in part (a) and part (b) are not equal.

(d) \( \lim_{t \to 2^-} g(t) = 2 \) (e) \( \lim_{t \to 2^+} g(t) = 0 \)

(f) \( \lim_{t \to 2} g(t) \) does not exist because the limits in part (d) and part (e) are not equal.

(g) \( g(2) = 1 \) (h) \( \lim_{t \to 4} g(t) = 3 \)

9. (a) \( \lim_{x \to -7} f(x) = -\infty \) (b) \( \lim_{x \to -3} f(x) = \infty \) (c) \( \lim_{x \to -4} f(x) = \infty \)

(d) \( \lim_{x \to -6} f(x) = -\infty \) (e) \( \lim_{x \to -6^+} f(x) = \infty \)

(f) The equations of the vertical asymptotes are \( x = -7 \), \( x = -3 \), \( x = 0 \), and \( x = 6 \).

11. (a) \( \lim_{x \to 0^-} f(x) = 1 \)

(b) \( \lim_{x \to 0^+} f(x) = 0 \)

(c) \( \lim_{x \to 0} f(x) \) does not exist because the limits in part (a) and part (b) are not equal.

13. \( \lim_{x \to 1^-} f(x) = 2, \lim_{x \to 1^+} f(x) = -2, f(1) = 2 \)

15. \( \lim_{x \to 3^-} f(x) = 4, \lim_{x \to 3^+} f(x) = -2, \lim_{x \to 2} f(x) = 2, f(3) = 3, f(-2) = 1 \)
17. For \( f(x) = \frac{x^2 - 2x}{x^2 - x - 2} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.714286</td>
<td>1.9</td>
<td>0.655172</td>
</tr>
<tr>
<td>2.1</td>
<td>0.677419</td>
<td>1.95</td>
<td>0.661017</td>
</tr>
<tr>
<td>2.05</td>
<td>0.672131</td>
<td>1.99</td>
<td>0.665552</td>
</tr>
<tr>
<td>2.01</td>
<td>0.667774</td>
<td>1.995</td>
<td>0.666110</td>
</tr>
<tr>
<td>2.005</td>
<td>0.667221</td>
<td>1.999</td>
<td>0.666556</td>
</tr>
<tr>
<td>2.001</td>
<td>0.666778</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 2} \frac{x^2 - 2x}{x^2 - x - 2} = 0.6 = \frac{3}{5} \).

19. For \( f(x) = \frac{e^x - 1 - x}{x^2} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.718282</td>
<td>-1</td>
<td>0.367879</td>
</tr>
<tr>
<td>0.5</td>
<td>0.594885</td>
<td>-0.5</td>
<td>0.426123</td>
</tr>
<tr>
<td>0.1</td>
<td>0.517092</td>
<td>-0.1</td>
<td>0.483742</td>
</tr>
<tr>
<td>0.05</td>
<td>0.508439</td>
<td>-0.05</td>
<td>0.497177</td>
</tr>
<tr>
<td>0.01</td>
<td>0.501671</td>
<td>-0.01</td>
<td>0.498337</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = 0.5 = \frac{1}{2} \).

21. For \( f(x) = \frac{\sqrt{x + 4} - 2}{x} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.236068</td>
<td>-1</td>
<td>0.267949</td>
</tr>
<tr>
<td>0.5</td>
<td>0.242641</td>
<td>-0.5</td>
<td>0.258343</td>
</tr>
<tr>
<td>0.1</td>
<td>0.248457</td>
<td>-0.1</td>
<td>0.251582</td>
</tr>
<tr>
<td>0.05</td>
<td>0.249224</td>
<td>-0.05</td>
<td>0.250786</td>
</tr>
<tr>
<td>0.01</td>
<td>0.249844</td>
<td>-0.01</td>
<td>0.250156</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x} = 0.25 = \frac{1}{4} \).

23. For \( f(x) = \frac{x^6 - 1}{x^{10} - 1} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.183369</td>
<td>1.1</td>
<td>0.484119</td>
</tr>
<tr>
<td>1.05</td>
<td>0.540783</td>
<td>1.01</td>
<td>0.588022</td>
</tr>
<tr>
<td>1.001</td>
<td>0.598800</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 1} \frac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5} \).

25. \( \lim_{x \to -3^+} \frac{x + 2}{x + 3} = -\infty \) since the numerator is negative and the denominator approaches 0 from the positive side as \( x \to -3^+ \).

27. \( \lim_{x \to 1} \frac{2 - x}{(x - 1)^2} = \infty \) since the numerator is positive and the denominator approaches 0 through positive values as \( x \to 1 \).

29. Let \( t = x^2 - 9 \). Then as \( x \to 3^+ , t \to 0^+ \), and \( \lim_{x \to 3^+} \ln(x^2 - 9) = \lim_{t \to 0^+} \ln t = -\infty \) by (3).

31. \( \lim_{x \to 2\pi^-} x \csc x = \lim_{x \to 2\pi^-} \frac{x}{\sin x} = -\infty \) since the numerator is positive and the denominator approaches 0 through negative values as \( x \to 2\pi^- \).

33. (a) \( f(x) = \frac{1}{x^3 - 1} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.42</td>
<td>0.99999933333</td>
<td>33.33333333</td>
</tr>
<tr>
<td>1.1</td>
<td>3.02</td>
<td>1.00000000</td>
<td>33.33333333</td>
</tr>
<tr>
<td>1.01</td>
<td>33.0</td>
<td>0.9999999999</td>
<td>33.33333333</td>
</tr>
<tr>
<td>0.999</td>
<td>333.0</td>
<td>0.9999999999</td>
<td>33.33333333</td>
</tr>
<tr>
<td>0.99</td>
<td>333.7</td>
<td>0.9999999999</td>
<td>33.33333333</td>
</tr>
<tr>
<td>0.9</td>
<td>333.7</td>
<td>0.9999999999</td>
<td>33.33333333</td>
</tr>
</tbody>
</table>

From these calculations, it seems that
\( \lim_{x \to 1^-} f(x) = -\infty \) and \( \lim_{x \to 1^+} f(x) = \infty \).
(b) If \( x \) is slightly smaller than 1, then \( x^3 - 1 \) will be a negative number close to 0, and the reciprocal of \( x^3 - 1 \), that is, \( f(x) \), will be a negative number with large absolute value. So \( \lim_{x \to 1^-} f(x) = -\infty \).

If \( x \) is slightly larger than 1, then \( x^3 - 1 \) will be a small positive number, and its reciprocal, \( f(x) \), will be a large positive number. So \( \lim_{x \to 1^+} f(x) = \infty \).

(c) It appears from the graph of \( f \) that
\[
\lim_{x \to 1^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \infty.
\]

35. (a) Let \( h(x) = (1 + x)^{1/x} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.001</td>
<td>2.71964</td>
</tr>
<tr>
<td>-0.0001</td>
<td>2.71842</td>
</tr>
<tr>
<td>-0.00001</td>
<td>2.71830</td>
</tr>
<tr>
<td>-0.000001</td>
<td>2.71828</td>
</tr>
<tr>
<td>0.000001</td>
<td>2.71828</td>
</tr>
<tr>
<td>0.0001</td>
<td>2.71827</td>
</tr>
<tr>
<td>0.001</td>
<td>2.71815</td>
</tr>
<tr>
<td>0.01</td>
<td>2.71692</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 0} (1 + x)^{1/x} \approx 2.71828 \), which is approximately \( e \).

In Section 3.6 we will see that the value of the limit is exactly \( e \).

37. For \( f(x) = x^2 - (2^2/1000) \):

(a) \[
\begin{array}{c|c}
 x & f(x) \\
---&---
 1 & 0.998000 \\
 0.8 & 0.638259 \\
 0.6 & 0.358484 \\
 0.4 & 0.158680 \\
 0.2 & 0.038851 \\
 0.1 & 0.008928 \\
 0.05 & 0.001465 \\
\end{array}
\]

It appears that \( \lim_{x \to 0} f(x) = 0 \).

(b) \[
\begin{array}{c|c}
 x & f(x) \\
---&---
 0.04 & 0.000572 \\
 0.02 & -0.000614 \\
 0.01 & -0.000907 \\
 0.005 & -0.000978 \\
 0.003 & -0.000993 \\
 0.001 & -0.001000 \\
\end{array}
\]

It appears that \( \lim_{x \to 0} f(x) = -0.001 \).
CHAPTER 2 LIMITS AND DERIVATIVES

39. No matter how many times we zoom in toward the origin, the graphs of \( f(x) = \sin(\pi/x) \) appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as \( x \to 0 \).

41. There appear to be vertical asymptotes of the curve \( y = \tan(2\sin x) \) at \( x \approx \pm 0.90 \) and \( x \approx \pm 2.24 \). To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at \( x = \frac{\pi}{2} + \pi n \). Thus, we must have \( 2\sin x = \frac{\pi}{2} + \pi n \), or equivalently, \( \sin x = \frac{\pi}{2} + \frac{\pi}{4} n \). Since \( -1 \leq \sin x \leq 1 \), we must have \( \sin x = \pm \frac{\pi}{4} \) and so \( x = \pm \sin^{-1} \frac{\pi}{4} \) (corresponding to \( x \approx \pm 0.90 \)). Just as \( 150^\circ \) is the reference angle for \( 30^\circ \), \( \pi - \sin^{-1} \frac{\pi}{4} \) is the reference angle for \( \sin^{-1} \frac{\pi}{4} \). So \( x = \pm (\pi - \sin^{-1} \frac{\pi}{4}) \) are also equations of vertical asymptotes (corresponding to \( x \approx \pm 2.24 \)).

2.3 Calculating Limits Using the Limit Laws

1. (a) \[ \lim_{x \to 2} [f(x) + 5g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} [5g(x)] \quad \text{[Limit Law 1]} \]
   \[ = \lim_{x \to 2} f(x) + 5 \lim_{x \to 2} g(x) \quad \text{[Limit Law 3]} \]
   \[ = 4 + 5(-2) = -6 \]

(b) \[ \lim_{x \to 2} [g(x)]^3 = \left[ \lim_{x \to 2} g(x) \right]^3 \quad \text{[Limit Law 6]} \]
   \[ = (-2)^3 = -8 \]

(c) \[ \lim_{x \to 2} \sqrt{f(x)} = \sqrt{\lim_{x \to 2} f(x)} \quad \text{[Limit Law 11]} \]
   \[ = \sqrt{4} = 2 \]
(4) \[ \lim_{x \to 4} \frac{3f(x)}{g(x)} = \frac{\lim_{x \to 4} [3f(x)]}{\lim_{x \to 4} g(x)} \quad \text{[Limit Law 5]} \]

\[ = \frac{3 \lim_{x \to 4} f(x)}{\lim_{x \to 4} g(x)} \quad \text{[Limit Law 3]} \]

\[ = \frac{3(4)}{-2} = -6 \]

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, \[ \lim_{x \to 2} \frac{g(x)}{h(x)}, \] does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) \[ \lim_{x \to 2} \frac{g(x)h(x)}{f(x)} = \frac{\lim_{x \to 2} [g(x)h(x)]}{\lim_{x \to 2} f(x)} \quad \text{[Limit Law 5]} \]

\[ = \frac{\lim_{x \to 2} g(x) \cdot \lim_{x \to 2} h(x)}{\lim_{x \to 2} f(x)} \quad \text{[Limit Law 4]} \]

\[ = \frac{-2 \cdot 0}{4} = 0 \]

3. \[ \lim_{x \to -2} (3x^4 + 2x^2 - x + 1) = \lim_{x \to -2} 3x^4 + \lim_{x \to -2} 2x^2 - \lim_{x \to -2} x + \lim_{x \to -2} 1 \quad \text{[Limit Laws 1 and 2]} \]

\[ = 3 \lim_{x \to -2} x^4 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} x + \lim_{x \to -2} 1 \quad \text{[3]} \]

\[ = 3(-2)^4 + 2(-2)^2 - (-2) + 1 \quad \text{[9, 8, and 7]} \]

\[ = 48 + 8 + 2 + 1 = 59 \]

5. \[ \lim_{x \to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3) = \lim_{x \to 8} (1 + \sqrt[3]{x}) \cdot \lim_{x \to 8} (2 - 6x^2 + x^3) \quad \text{[Limit Law 4]} \]

\[ = \left( \lim_{x \to 8} 1 + \sqrt[3]{x} \right) \cdot \left( \lim_{x \to 8} 2 - 6 \lim_{x \to 8} x^2 + \lim_{x \to 8} x^3 \right) \quad \text{[1, 2, and 3]} \]

\[ = (1 + \sqrt[3]{8}) \cdot (2 - 6 \cdot 8^2 + 8^3) \quad \text{[7, 10, 9]} \]

\[ = (3)(130) = 390 \]

7. \[ \lim_{x \to 1} \left( \frac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3 = \left( \lim_{x \to 1} \frac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3 \quad \text{[6]} \]

\[ = \left[ \lim_{x \to 1} \frac{1 + 3x}{1 + 4x^2 + 3x^4} \right]^3 \quad \text{[5]} \]

\[ = \left[ \lim_{x \to 1} \frac{1 + 3 \lim_{x \to 1} x}{\lim_{x \to 1} 1 + 4 \lim_{x \to 1} x^2 + 3 \lim_{x \to 1} x^4} \right]^3 \quad \text{[2, 1, and 3]} \]

\[ = \left[ \frac{1 + 3(1)}{1 + 4(1)^2 + 3(1)^4} \right]^3 = \left[ \frac{4}{8} \right]^3 \quad \text{= \left( \frac{1}{2} \right)^3 = \frac{1}{8} \quad [7, 8, and 9]} \]

9. \[ \lim_{x \to 4} \sqrt{16 - x^2} = \sqrt{\lim_{x \to 4} (16 - x^2)} \quad \text{[11]} \]

\[ = \sqrt{\lim_{x \to 4} 16 - \lim_{x \to 4} x^2} \quad \text{[2]} \]

\[ = \sqrt{16 - (4)^2} = 0 \quad \text{[7 and 9]} \]
By the formula for the sum of cubes, we have

\[
\lim_{x \to 2} \frac{x^2 - x + 6}{x - 2} \]

does not exist since \( x - 2 \to 0 \) but \( x^2 - x + 6 \to 8 \) as \( x \to 2 \).

By the formula for the sum of cubes, we have

\[
\lim_{h \to 0} \frac{(4 + h)^3 - 16}{h} = \lim_{h \to 0} \frac{(16 + 8h + h^2) - 16}{h} = \lim_{h \to 0} \frac{8h + h^2}{h} = \lim_{h \to 0} \frac{h(8 + h)}{h} = \lim_{h \to 0} (8 + h) = 8 + 0 = 8
\]

19. By the formula for the sum of cubes, we have

\[
\lim_{x \to -2} \frac{x + 2}{x^3 + 8} = \lim_{x \to -2} \frac{x + 2}{(x + 2)(x^2 - 2x + 4)} = \lim_{x \to -2} \frac{1}{x^2 - 2x + 4} = \frac{1}{4 + 4 + 4} = \frac{1}{12}
\]

21. \[
\lim_{t \to 3} \frac{9 - t}{3 - \sqrt[9]{t}} = \lim_{t \to 9} \frac{(3 + \sqrt[3]{t})(3 - \sqrt[3]{t})}{3 - \sqrt[9]{t}} = \lim_{t \to 9} (3 + \sqrt[9]{t}) = 3 + \sqrt[9]{9} = 6
\]

23. \[
\lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7} = \lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7} \cdot \frac{\sqrt{x + 2} + 3}{\sqrt{x + 2} + 3} = \lim_{x \to 7} \frac{(x + 2) - 9}{x - 7}(\sqrt{x + 2} + 3)
\]

25. \[
\lim_{x \to -4} \frac{4 - \sqrt[4]{x}}{4 + x} = \lim_{x \to -4} \frac{4 - \sqrt[4]{x}}{4 + x} \cdot \frac{4x}{4x} = \lim_{x \to -4} \frac{x + 4}{4x} = \lim_{x \to -4} \frac{x + 4}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}
\]

27. \[
\lim_{x \to 16} \frac{4 - \sqrt[4]{x}}{16x - x^2} = \lim_{x \to 16} \frac{(4 - \sqrt[4]{x})(4 + \sqrt[4]{x})}{16x - x^2} = \lim_{x \to 16} \frac{16 - x}{x(16 - x)(4 + \sqrt[4]{x})} = \lim_{x \to 16} \frac{1}{x(4 + \sqrt[4]{x})} = \frac{1}{16(4 + \sqrt[4]{16})} = \frac{1}{168} = \frac{1}{128}
\]

29. \[
\lim_{t \to 0} \frac{1}{t \sqrt[1+t]{t}} = \lim_{t \to 0} \frac{1}{t \sqrt[1+t]{t}} = \lim_{t \to 0} \frac{(1 - \sqrt{1 + t})(1 + \sqrt{1 + t})}{t t \sqrt{1 + t} (1 + \sqrt{1 + t})} = \lim_{t \to 0} \frac{-t}{t \sqrt{1 + t} (1 + \sqrt{1 + t})} = \lim_{t \to 0} \frac{-1}{\sqrt{1 + t} (1 + \sqrt{1 + t})} = \frac{-1}{\sqrt{1 + 0} (1 + \sqrt{1 + 0})} = \frac{-1}{2}
\]

31. (a) \[
\lim_{x \to -1} \frac{x}{\sqrt{1 + 3x} - 1} \approx \frac{2}{3}
\]

(b) \[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
-0.001 & 0.6661663 \\
-0.0001 & 0.6661667 \\
-0.00001 & 0.6666167 \\
-0.000001 & 0.6666667 \\
0.000001 & 0.6666662 \\
0.000001 & 0.6666672 \\
0.00001 & 0.6666717 \\
0.001 & 0.6667167 \\
0.01 & 0.6671663 \\
\hline
\end{array}
\]

The limit appears to be \( \frac{2}{3} \).
(c) \[ \lim_{x \to 0} \left( \frac{x}{\sqrt{1 + 3x} - 1} \right) = \lim_{x \to 0} \frac{x(\sqrt{1 + 3x} - 1 + 1)}{(\sqrt{1 + 3x} - 1)(\sqrt{1 + 3x} + 1)} = \lim_{x \to 0} \frac{x}{3x} = \frac{1}{3} \lim_{x \to 0} (\sqrt{1 + 3x} + 1) \]

[Limit Law 3]

\[ = \frac{1}{3} \left( \lim_{x \to 0} (1 + 3x) + \lim_{x \to 0} 1 \right) \]

[1 and 11]

\[ = \frac{1}{3} \left( \lim_{x \to 0} 1 + 3 \lim_{x \to 0} x + 1 \right) \]

[1, 3, and 7]

\[ = \frac{1}{3} (\sqrt{1} + 3 \cdot 0 + 1) \]

[7 and 8]

\[ = \frac{1}{3} (1 + 1) = \frac{2}{3} \]

33. Let \( f(x) = -x^2 \), \( g(x) = x^2 \cos 20\pi x \) and \( h(x) = x^2 \). Then

\[ -1 \leq \cos 20\pi x \leq 1 \Rightarrow -x^2 \leq x^2 \cos 20\pi x \leq x^2 \Rightarrow f(x) \leq g(x) \leq h(x). \]

So since \( \lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \), by the Squeeze Theorem we have

\[ \lim_{x \to 0} g(x) = 0. \]

35. We have \( \lim_{x \to 4} (4x - 9) = 4(4) - 9 = 7 \) and \( \lim_{x \to 4} (x^2 - 4x + 7) = 4^2 - 4(4) + 7 = 7. \) Since \( 4x - 9 \leq f(x) \leq x^2 - 4x + 7 \) for \( x \geq 0 \), \( \lim_{x \to 4} f(x) = 7 \) by the Squeeze Theorem.

37. \(-1 \leq \cos(2/x) \leq 1 \Rightarrow -x^4 \leq x^4 \cos(2/x) \leq x^4. \) Since \( \lim_{x \to 0} (-x^4) = 0 \) and \( \lim_{x \to 0} x^4 = 0 \), we have

\[ \lim_{x \to 0} [x^4 \cos(2/x)] = 0 \] by the Squeeze Theorem.

39. \[ |x - 3| = \begin{cases} 
  x - 3 & \text{if } x - 3 \geq 0 \\
  -(x - 3) & \text{if } x - 3 < 0
\end{cases} = \begin{cases} 
  x - 3 & \text{if } x \geq 3 \\
  -x + 3 & \text{if } x < 3
\end{cases} \]

Thus, \( \lim_{x \to 3^+} (2x + |x - 3|) = \lim_{x \to 3^+} (2x + x - 3) = \lim_{x \to 3^+} (3x - 3) = 3(3) - 3 = 6 \) and

\[ \lim_{x \to 3^-} (2x + |x - 3|) = \lim_{x \to 3^-} (2x + 3 - x) = \lim_{x \to 3^-} (x + 3) = 3 + 3 = 6. \] Since the left and right limits are equal,

\[ \lim_{x \to 3} (2x + |x - 3|) = 6. \]

41. \[ |2x^3 - x^2| = |x^2(2x - 1)| = |x^2| \cdot |2x - 1| = x^2 |2x - 1| \]

\[ |2x - 1| = \begin{cases} 
  2x - 1 & \text{if } 2x - 1 \geq 0 \\
  -(2x - 1) & \text{if } 2x - 1 < 0
\end{cases} = \begin{cases} 
  2x - 1 & \text{if } x \geq 0.5 \\
  -(2x - 1) & \text{if } x < 0.5
\end{cases} \]

Thus, \( |2x^3 - x^2| = x^2 |-(2x - 1)| \) for \( x < 0.5. \)

\[ = \lim_{x \to 0.5^-} \frac{2x - 1}{|2x^3 - x^2|} = \lim_{x \to 0.5^-} \frac{2x - 1}{(2x - 1)} \]

\[ = \lim_{x \to 0.5^-} \frac{-1}{(0.5)^2} = \frac{-1}{0.25} = -4. \]
43. Since \(|x| = -x\) for \(x < 0\), we have \(\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{-x}\right) = \lim_{x \to 0^-} \frac{2}{x}\), which does not exist since the denominator approaches 0 and the numerator does not.

45. (a) 

(b) (i) Since \(\text{sgn} \, x = 1\) for \(x > 0\), \(\lim_{x \to 0^+} \text{sgn} \, x = \lim_{x \to 0^+} 1 = 1\).

(ii) Since \(\text{sgn} \, x = -1\) for \(x < 0\), \(\lim_{x \to 0^-} \text{sgn} \, x = \lim_{x \to 0^-} -1 = -1\).

(iii) Since \(\lim_{x \to 0^-} \text{sgn} \, x \neq \lim_{x \to 0^+} \text{sgn} \, x\), \(\lim_{x \to 0} \text{sgn} \, x\) does not exist.

(iv) Since \(|\text{sgn} \, x| = 1\) for \(x \neq 0\), \(\lim_{x \to 0} |\text{sgn} \, x| = \lim_{x \to 0} 1 = 1\).

47. (a) (i) \(\lim_{x \to 1^+} F(x) = \lim_{x \to 1^+} \frac{x^2 - 1}{|x - 1|} = \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^+} (x + 1) = 2\) (c) 

(ii) \(\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} \frac{x^2 - 1}{|x - 1|} = \lim_{x \to 1^-} \frac{x^2 - 1}{-(x - 1)} = \lim_{x \to 1^-} -(x + 1) = -2\)

(b) No, \(\lim_{x \to 1^-} F(x)\) does not exist since \(\lim_{x \to 1^+} F(x) \neq \lim_{x \to 1^-} F(x)\).

49. (a) (i) \([x] = -2\) for \(-2 \leq x < -1\), so \(\lim_{x \to -2^+} [x] = \lim_{x \to -2^+} (-2) = -2\)

(ii) \([x] = -3\) for \(-3 \leq x < -2\), so \(\lim_{x \to -2^+} [x] = \lim_{x \to -2^+} (-3) = -3\).

The right and left limits are different, so \(\lim_{x \to -2} [x]\) does not exist.

(iii) \([x] = -3\) for \(-3 \leq x < -2\), so \(\lim_{x \to -2^-} [x] = \lim_{x \to -2^-} (-3) = -3\).

(b) (i) \([x] = n - 1\) for \(n - 1 \leq x < n\), so \(\lim_{x \to n^-} [x] = \lim_{x \to n^-} (n - 1) = n - 1\).

(ii) \([x] = n\) for \(n \leq x < n + 1\), so \(\lim_{x \to n^+} [x] = \lim_{x \to n^+} n = n\).

(c) \(\lim_{x \to a} [x]\) exists \(\iff\ a\) is not an integer.

51. The graph of \(f(x) = [x] + [−x]\) is the same as the graph of \(g(x) = −1\) with holes at each integer, since \(f(a) = 0\) for any integer \(a\). Thus, \(\lim_{x \to 2^-} f(x) = -1\) and \(\lim_{x \to 2^+} f(x) = -1\), so \(\lim_{x \to 2} f(x) = -1\). However, \(f(2) = [2] + [-2] = 2 + (-2) = 0\), so \(\lim_{x \to 2} f(x) \neq f(2)\).

53. Since \(p(x)\) is a polynomial, \(p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\). Thus, by the Limit Laws,

\[
\lim_{x \to a} p(x) = \lim_{x \to a} (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \cdots + a_n \lim_{x \to a} x^n
\]

\[
= a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n = p(a)
\]

Thus, for any polynomial \(p\), \(\lim_{x \to a} p(x) = p(a)\).
55. \[ \lim_{x \to 1} [f(x) - 8] = \lim_{x \to 1} \left( \frac{f(x) - 8}{x - 1} \cdot (x - 1) \right) = \lim_{x \to 1} \frac{f(x) - 8}{x - 1} \cdot \lim_{x \to 1} (x - 1) = 10 \cdot 0 = 0. \]

Thus, \( \lim_{x \to 1} f(x) = \lim_{x \to 1} [f(x) - 8] + \lim_{x \to 1} 8 = 0 + 8 = 8. \)

Note: The value of \( \lim_{x \to 1} \frac{f(x) - 8}{x - 1} \) does not affect the answer since it’s multiplied by 0. What’s important is that exists.

57. Observe that \( 0 \leq f(x) \leq x^2 \) for all \( x \), and \( \lim_{x \to 0} 0 = \lim_{x \to 0} x^2. \) So, by the Squeeze Theorem, \( \lim_{x \to 0} f(x) = 0. \)

59. Let \( f(x) = H(x) \) and \( g(x) = 1 - H(x) \), where \( H \) is the Heaviside function defined in Exercise 1.3.57.

Thus, either \( f \) or \( g \) is 0 for any value of \( x \). Then \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) do not exist, but \( \lim_{x \to 0} [f(x)g(x)] = \lim_{x \to 0} 0 = 0. \)

61. Since the denominator approaches 0 as \( x \to -2 \), the limit will exist only if the numerator also approaches 0 as \( x \to -2. \) In order for this to happen, we need \( \lim_{x \to -2} (3x^2 + ax + a + 3) = 0 \) \( \iff \)

\[ 3(-2)^2 + a(-2) + a + 3 = 0 \iff 12 - 2a + a + 3 = 0 \iff a = 15. \] With \( a = 15, \) the limit becomes

\[ \lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \to -2} \frac{3(x + 2)(x + 3)}{(x - 1)(x + 2)} = \lim_{x \to -2} \frac{3(x + 3)}{x - 1} = \frac{3(-2 + 3)}{-2 - 1} = \frac{-3}{-3} = -1. \]

2.4 The Precise Definition of a Limit

1. On the left side of \( x = 2, \) we need \( |x - 2| < \frac{10}{4} - 2 = \frac{4}{2}. \) On the right side, we need \( |x - 2| < \frac{10}{4} - 2 = \frac{4}{2}. \) For both of these conditions to be satisfied at once, we need the more restrictive of the two to hold, that is, \( |x - 2| < \frac{4}{2}. \) So we can choose \( \delta = \frac{4}{2}, \) or any smaller positive number.

3. The leftmost question mark is the solution of \( \sqrt{x} = 1.6 \) and the rightmost, \( \sqrt{x} = 2.4. \) So the values are \( 1.6^2 = 2.56 \) and \( 2.4^2 = 5.76. \) On the left side, we need \( |x - 4| < |2.56 - 4| = 1.44. \) On the right side, we need \( |x - 4| < |5.76 - 4| = 1.76. \) To satisfy both conditions, we need the more restrictive condition to hold—namely, \( |x - 4| < 1.44. \) Thus, we can choose \( \delta = 1.44, \) or any smaller positive number.

5. From the graph, we find that \( \tan x = 0.8 \) when \( x \approx 0.675, \) so

\[ \frac{\pi}{4} - \delta_1 \approx 0.675 \Rightarrow \delta_1 \approx \frac{\pi}{4} - 0.675 \approx 0.1106. \]

Also, \( \tan x = 1.2 \)

when \( x \approx 0.876, \) so \( \frac{\pi}{4} + \delta_2 \approx 0.876 \Rightarrow \delta_2 \approx 0.876 - \frac{\pi}{4} \approx 0.0906. \)

Thus, we choose \( \delta = 0.0906 \) (or any smaller positive number) since this is the smaller of \( \delta_1 \) and \( \delta_2. \)
7. For \( \varepsilon = 1 \), the definition of a limit requires that we find \( \delta \) such that \( |(4 + x - 3x^3) - 2| < 1 \iff 1 < 4 + x - 3x^3 < 3 \) whenever \( 0 < |x - 1| < \delta \). If we plot the graphs of \( y = 1, y = 4 + x - 3x^3 \) and \( y = 3 \) on the same screen, we see that we need \( 0.86 \leq x \leq 1.11 \). So since \( |1 - 0.86| = 0.14 \) and \( |1 - 1.11| = 0.11 \), we choose \( \delta = 0.11 \) (or any smaller positive number). For \( \varepsilon = 0.1 \), we must find \( \delta \) such that \( |(4 + x - 3x^3) - 2| < 0.1 \iff 1.9 < 4 + x - 3x^3 < 2.1 \) whenever \( 0 < |x - 1| < \delta \). From the graph, we see that we need \( 0.988 \leq x \leq 1.012 \). So since \( |1 - 0.988| = 0.012 \) and \( |1 - 1.012| = 0.012 \), we choose \( \delta = 0.012 \) (or any smaller positive number) for the inequality to hold.

From the graph, we find that \( y = \tan^2 x = 1000 \) when \( x \approx 1.539 \) and \( x \approx 1.602 \) for \( x \) near \( \frac{\pi}{4} \). Thus, we get \( \delta \approx 1.602 - \frac{\pi}{4} \approx 0.031 \) for \( M = 1000 \).

From the graph, we find that \( y = \tan^2 x = 10,000 \) when \( x \approx 1.561 \) and \( x \approx 1.581 \) for \( x \) near \( \frac{\pi}{4} \). Thus, we get \( \delta \approx 1.581 - \frac{\pi}{4} \approx 0.010 \) for \( M = 10,000 \).

11. (a) \( A = \pi r^2 \) and \( A = 1000 \, \text{cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow r = \sqrt{\frac{1000}{\pi}} \ (r > 0) \approx 17.8412 \, \text{cm} \).

(b) \( |A - 1000| \leq 5 \Rightarrow -5 \leq \pi r^2 - 1000 \leq 5 \Rightarrow 1000 - 5 \leq \pi r^2 \leq 1000 + 5 \Rightarrow \sqrt{\frac{995}{\pi}} \leq r \leq \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \leq r \leq 17.8858 \). \( \sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466 \) and \( \sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455 \). So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm² of 1000.

(c) \( x \) is the radius, \( f(x) \) is the area, \( a \) is the target radius given in part (a), \( L \) is the target area (1000), \( \varepsilon \) is the tolerance in the area (5), and \( \delta \) is the tolerance in the radius given in part (b).

13. (a) \( |4x - 8| = 4 |x - 2| < 0.1 \iff |x - 2| < 0.1, \) so \( \delta = \frac{0.1}{4} = 0.025 \).

(b) \( |4x - 8| = 4 |x - 2| < 0.01 \iff |x - 2| < 0.01, \) so \( \delta = \frac{0.01}{4} = 0.0025 \).
15. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 1| < \delta \), then
\[
|2x + 3 - 5| < \varepsilon. \quad \text{But } |(2x + 3) - 5| < \varepsilon \iff |2x - 2| < \varepsilon \iff 2|x - 1| < \varepsilon \iff |x - 1| < \varepsilon/2.
\]
So if we choose \( \delta = \varepsilon/2 \), then \( 0 < |x - 1| < \delta \implies |(2x + 3) - 5| < \varepsilon \). Thus, \( \lim_{x \to 1} (2x + 3) = 5 \) by the definition of a limit.

17. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - (-3)| < \delta \), then
\[
|(1 - 4x) - 13| < \varepsilon. \quad \text{But } |(1 - 4x) - 13| < \varepsilon \iff |-4| |x + 3| < \varepsilon \iff |x - (-3)| < \varepsilon/4.
\]
So if we choose \( \delta = \varepsilon/4 \), then \( 0 < |x - (-3)| < \delta \implies |(1 - 4x) - 13| < \varepsilon \). Thus, \( \lim_{x \to -3} (1 - 4x) = 13 \) by the definition of a limit.

19. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 3| < \delta \), then
\[
\left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon \iff \frac{1}{5} |x - 3| < \varepsilon \iff |x - 3| < 5\varepsilon.
\]
So choose \( \delta = 5\varepsilon \). Then \( 0 < |x - 3| < \delta \implies |x - 3| < 5\varepsilon \implies \left| \frac{x - 3}{5} \right| < \varepsilon \implies \left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon \). By the definition of a limit, \( \lim_{x \to 3} \frac{x}{5} = \frac{3}{5} \).

21. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then
\[
\left| \frac{x^2 + x - 6}{x - 2} - 5 \right| < \varepsilon \iff \left| \frac{(x + 3)(x - 2)}{x - 2} - 5 \right| < \varepsilon \iff \left| x + 3 - 5 \right| < \varepsilon \iff |x - 2| < \varepsilon. \quad \text{So choose } \delta = \varepsilon.
\]
Then \( 0 < |x - 2| < \delta \implies |x - 2| < \varepsilon \implies |x + 3 - 5| < \varepsilon \implies \left| \frac{(x + 3)(x - 2)}{x - 2} - 5 \right| < \varepsilon \iff |x - 2| < \varepsilon \). By the definition of a limit, \( \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = 5 \).

23. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \), then \( |x - a| < \varepsilon \). So \( \delta = \varepsilon \) will work.

25. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \), then \( |x^2 - 0| < \varepsilon \iff x^2 < \varepsilon \iff |x| < \sqrt{\varepsilon}. \quad \text{Take } \delta = \sqrt{\varepsilon}.
\]
Then \( 0 < |x - 0| < \delta \implies |x^2 - 0| < \varepsilon \). Thus, \( \lim_{x \to 0} x^2 = 0 \) by the definition of a limit.

27. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \), then \( |x - 0| < \varepsilon \). But \( |x| = |x| \). So this is true if we pick \( \delta = \varepsilon \).

Thus, \( \lim_{x \to 0} |x| = 0 \) by the definition of a limit.
29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|x^2 - 4x + 5 - 1| < \varepsilon \iff |x^2 - 4x + 4| < \varepsilon \iff |(x - 2)^2| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 2| < \delta \implies |x - 2| < \sqrt{\varepsilon} \implies |(x - 2)^2| < \varepsilon$. Thus, 
\[
\lim_{x \to 2} (x^2 - 4x + 5) = 1 \text{ by the definition of a limit.}
\]

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|x^2 - 1 - 3| < \varepsilon$ or upon simplifying we need $|x^2 - 4| < \varepsilon$ whenever $0 < |x + 2| < \delta$. Notice that if $|x + 2| < 1$, then $-1 < x + 2 < 1 \implies -5 < x - 2 < -3 \implies |x - 2| < 5$. Therefore $\delta = \min \{\varepsilon/5, 1\}$. Then $0 < |x + 2| < \delta \implies |x - 2| < 5$ and $|x + 2| < \varepsilon/5$, so $|x^2 - 1 - 3| = |(x + 2)(x - 2)| = |x + 2||x - 2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, 
\[
\lim_{x \to -2} (x^2 - 1) = 3.
\]

33. Given $\varepsilon > 0$, we let $\delta = \min \{2, \varepsilon/8\}$. If $0 < |x - 1| < 2$, then $|x - 3| < 2 \implies -2 < x - 3 < 2 \implies 4 < x + 3 < 8 \implies |x + 3| < 8$. Also $|x - 3| < \varepsilon/8$, so $|x^2 - 9| = |x + 3||x - 3| < 8 \cdot \varepsilon/8 = \varepsilon$. Thus, $\lim_{x \to 3} x^2 = 9$.

35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take $\delta$ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - x_1 \approx 0.093$.

(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is
\[
x(\varepsilon) = \frac{(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2})^{2/3} - 12}{6 (216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2})^{1/3}}.
\]
Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093272342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

37. 1. Guessing a value for $\delta$ Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x - a| < \delta$. But
\[
|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \varepsilon \quad \text{(from the hint).}
\]
Now if we can find a positive constant $C$ such that $\sqrt{x} + \sqrt{a} > C$ then
\[
\frac{|x - a|}{\sqrt{x} + \sqrt{a}} < C \implies |x - a| < C \varepsilon, \text{ and we take } |x - a| < C \varepsilon.
\]
We can find this number by restricting $x$ to lie in some interval centered at $a$. If $|x - a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \implies \frac{1}{2}a < x < \frac{3}{2}a \implies \sqrt{x} + \sqrt{a} > \sqrt{\frac{3}{2}a + \sqrt{a}}$, and so
\[
C = \sqrt{\frac{3}{2}a + \sqrt{a}} \text{ is a suitable choice for the constant.}
\]
So $|x - a| < \left(\sqrt{\frac{3}{2}a + \sqrt{a}}\right)\varepsilon$. This suggests that we let
\[
\delta = \min \left\{\frac{1}{2}a, \left(\sqrt{\frac{3}{2}a + \sqrt{a}}\right)\varepsilon\right\}.
\]

2. Showing that $\delta$ works Given $\varepsilon > 0$, we let $\delta = \min \left\{\frac{1}{2}a, \left(\sqrt{\frac{3}{2}a + \sqrt{a}}\right)\varepsilon\right\}$. If $0 < |x - a| < \delta$, then
\[
|x - a| < \frac{1}{2}a \implies \sqrt{x} + \sqrt{a} > \sqrt{\frac{3}{2}a + \sqrt{a}} \text{ (as in part 1).}
\]
Also $|x - a| < \left(\sqrt{\frac{3}{2}a + \sqrt{a}}\right)\varepsilon$, so
\[
|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{\left(\sqrt{\frac{3}{2}a + \sqrt{a}}\right)^2}{\left(\sqrt{\frac{3}{2}a + \sqrt{a}}\right)^2} \varepsilon = \varepsilon.
\]
Therefore, $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ by the definition of a limit.
39. Suppose that \( \lim_{x \to 0} f(x) = L \). Given \( \varepsilon = \frac{1}{2} \), there exists \( \delta > 0 \) such that \( 0 < |x| < \delta \) \( \Rightarrow \) \( |f(x) - L| < \frac{1}{2} \). Take any rational number \( r \) with \( 0 < |r| < \delta \). Then \( f(r) = 0 \), so \( |0 - L| < \frac{1}{2} \), so \( L \leq |L| < \frac{1}{2} \). Now take any irrational number \( s \) with \( 0 < |s| < \delta \). Then \( f(s) = 1 \), so \( |1 - L| < \frac{1}{2} \). Hence, \( 1 - L < \frac{1}{2} \), so \( L > \frac{1}{2} \). This contradicts \( L < \frac{1}{2} \), so \( \lim_{x \to 0} f(x) \) does not exist.

41. \( \frac{1}{(x + 3)^4} > 10,000 \iff (x + 3)^4 < \frac{1}{10,000} \iff |x + 3| < \frac{1}{\sqrt[4]{10,000}} \iff |x - (-3)| < \frac{1}{10} \)

43. Given \( M < 0 \) we need \( \delta > 0 \) so that \( \ln x < M \) whenever \( 0 < x < \delta \); that is, \( x = e^{\ln x} < e^M \) whenever \( 0 < x < \delta \). This suggests that we take \( \delta = e^M \). If \( 0 < x < e^M \), then \( \ln x < \ln e^M = M \). By the definition of a limit, \( \lim_{x \to 0^+} \ln x = -\infty \).

2.5 Continuity

1. From Definition 1, \( \lim_{x \to 4} f(x) = f(4) \).

3. (a) The following are the numbers at which \( f \) is discontinuous and the type of discontinuity at that number: \(-4\) (removable), \(-2\) (jump), \(2\) (jump), \(4\) (infinite).

(b) \( f \) is continuous from the left at \(-2\) since \( \lim_{x \to -2^-} f(x) = f(-2) \). \( f \) is continuous from the right at 2 and 4 since \( \lim_{x \to 2^+} f(x) = f(2) \) and \( \lim_{x \to 4^+} f(x) = f(4) \). It is continuous from neither side at \(-4\) since \( f(-4) \) is undefined.

5. The graph of \( y = f(x) \) must have a discontinuity at \( x = 3 \) and must show that \( \lim_{x \to 3^-} f(x) = f(3) \).

7. (a) There are discontinuities at times \( t = 1, 2, 3, \) and \( 4 \). A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

9. Since \( f \) and \( g \) are continuous functions,
\[
\lim_{x \to 3^+} \left[ 2f(x) - g(x) \right] = 2 \lim_{x \to 3^+} f(x) - \lim_{x \to 3^+} g(x) \quad \text{[by Limit Laws 2 and 3]}
\]
\[
= 2f(3) - g(3) \quad \text{[by continuity of \( f \) and \( g \) at \( x = 3 \)]}
\]
\[
= 2 \cdot 5 - g(3) = 10 - g(3)
\]
Since it is given that \( \lim_{x \to 3} [2f(x) - g(x)] = 4 \), we have \( 10 - g(3) = 4 \), so \( g(3) = 6 \).
11. By Theorem 5, the polynomials
\[
\lim_{x \to -1} f(x) = \lim_{x \to -1} (x + 2x^3)^4 = \left( \lim_{x \to -1} x + 2 \lim_{x \to -1} x^3 \right)^4 = [-1 + 2(-1)^3]^4 = (-3)^4 = 81 = f(-1).
\]

By the definition of continuity, \( f \) is continuous at \( a = -1 \).

13. For \( a > 2 \), we have
\[
\lim_{x \to a} f(x) = \lim_{x \to a} \frac{2x + 3}{x - 2} = \frac{\lim_{x \to a} (2x + 3)}{\lim_{x \to a} (x - 2)} \quad \text{[Limit Law 5]}
\]
\[
= \frac{2 \lim_{x \to a} x + \lim_{x \to a} 3}{\lim_{x \to a} x - \lim_{x \to a} 2} \quad \text{[1, 2, and 3]}
\]
\[
= \frac{2a + 3}{a - 2} \quad \text{[7 and 8]}
\]
\[
= f(a)
\]

Thus, \( f \) is continuous at \( x = a \) for every \( a \) in \( (2, \infty) \); that is, \( f \) is continuous on \( (2, \infty) \).

15. \( f(x) = \ln |x - 2| \) is discontinuous at 2 since \( f(2) = \ln 0 \) is not defined.

17. \( f(x) = \begin{cases} 
    e^x & \text{if } x < 0 \\
    x^2 & \text{if } x \geq 0
\end{cases} \)

The left-hand limit of \( f \) at \( a = 0 \) is \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = 1 \). The right-hand limit of \( f \) at \( a = 0 \) is \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0 \). Since these limits are not equal, \( \lim_{x \to 0} f(x) \) does not exist and \( f \) is discontinuous at 0.

19. \( f(x) = \begin{cases} 
    \cos x & \text{if } x < 0 \\
    0 & \text{if } x = 0 \\
    1 - x^2 & \text{if } x > 0
\end{cases} \)

\[
\lim_{x \to 0} f(x) = 1, \text{ but } f(0) = 0 \neq 1, \text{ so } f \text{ is discontinuous at } 0.
\]

21. \( F(x) = \frac{x}{x^2 + 5x + 6} \) is a rational function. So by Theorem 5 (or Theorem 7), \( F \) is continuous at every number in its domain,
\[
\{ x \mid x^2 + 5x + 6 \neq 0 \} = \{ x \mid (x + 3)(x + 2) \neq 0 \} = \{ x \mid x \neq -3, -2 \} \text{ or } (-\infty, -3) \cup (-3, -2) \cup (-2, \infty).
\]

23. By Theorem 5, the polynomials \( x^2 \) and \( 2x - 1 \) are continuous on \( (-\infty, \infty) \). By Theorem 7, the root function \( \sqrt{x} \) is continuous on \( [0, \infty) \). By Theorem 9, the composite function \( \sqrt{2x - 1} \) is continuous on its domain, \( [\frac{1}{2}, \infty) \).

By part 1 of Theorem 4, the sum \( R(x) = x^2 + \sqrt{2x - 1} \) is continuous on \( [\frac{1}{2}, \infty) \).
25. By Theorem 7, the exponential function $e^{-5t}$ and the trigonometric function $\cos 2\pi t$ are continuous on $(-\infty, \infty)$.

By part 4 of Theorem 4, $L(t) = e^{-5t} \cos 2\pi t$ is continuous on $(-\infty, \infty)$.

27. By Theorem 5, the polynomial $t^4 - 1$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\ln x$ is continuous on its domain, $(0, \infty)$.

By Theorem 9, $\ln(t^4 - 1)$ is continuous on its domain, which is

$$\{ t \mid t^4 - 1 > 0 \} = \{ t \mid t^4 > 1 \} = \{ t \mid |t| > 1 \} = (-\infty, -1) \cup (1, \infty)$$

29. The function $y = \frac{1}{1 + e^{1/\sqrt{t}}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.

31. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x + 5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \frac{5 + \sqrt{x}}{\sqrt{5} + x}$ is continuous on $[0, \infty)$. Since $f$ is continuous at $x = 4$, $\lim_{x \to 4} f(x) = f(4) = \frac{7}{4}$.

33. Because $x^2 - x$ is continuous on $\mathbb{R}$, the composite function $f(x) = e^{x^2 - x}$ is continuous on $\mathbb{R}$, so

$$\lim_{x \to 1} f(x) = f(1) = e^{1-1} = e^0 = 1.$$  

35. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial $x^2$ on $(-\infty, 1)$, $f$ is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, $f$ is continuous on $(1, \infty)$. At $x = 1$, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x} = 1$. Thus, $f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, $f$ is continuous at $x = 1$.

We conclude that $f$ is continuous on $(-\infty, \infty)$.

37. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

$f$ is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1 + x^2) = 1$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2 - x) = 2$, so $f$ is discontinuous at 0. Since $f(0) = 1$, $f$ is continuous from the left at 0. Also, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x) = 0$, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2)^2 = 0$, and $f(2) = 0$, so $f$ is continuous at 2. The only number at which $f$ is discontinuous is 0.
39. \( f(x) = \begin{cases} 
 x + 2 & \text{if } x < 0 \\
 e^x & \text{if } 0 \leq x \leq 1 \\
 2 - x & \text{if } x > 1 
\end{cases} \)

\( f \) is continuous on \((-\infty, 0)\) and \((1, \infty)\) since on each of these intervals it is a polynomial; it is continuous on \((0, 1)\) since it is an exponential.

Now \( \lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^-}} (x + 2) = 2 \) and \( \lim_{{x \to 0^+}} f(x) = \lim_{{x \to 0^+}} e^x = 1 \), so \( f \) is discontinuous at 0. Since \( f(0) = 1 \), \( f \) is continuous from the right at 0. Also \( \lim_{{x \to 1^-}} f(x) = \lim_{{x \to 1^-}} e^x = e \) and \( \lim_{{x \to 1^+}} f(x) = \lim_{{x \to 1^+}} (2 - x) = 1 \), so \( f \) is discontinuous at 1. Since \( f(1) = e \), \( f \) is continuous from the left at 1.

41. \( f(x) = \begin{cases} 
 cx^2 + 2x & \text{if } x < 2 \\
 x^3 - cx & \text{if } x \geq 2 
\end{cases} \)

\( f \) is continuous on \((-\infty, 2)\) and \((2, \infty)\). Now \( \lim_{{x \to 2^-}} f(x) = \lim_{{x \to 2^-}} (cx^2 + 2x) = 4c + 4 \) and

\( \lim_{{x \to 2^+}} f(x) = \lim_{{x \to 2^+}} (x^3 - cx) = 8 - 2c. \)

So \( f \) is continuous \( \iff \) \( 4c + 4 = 8 - 2c \iff 6c = 4 \iff c = \frac{2}{3}. \) Thus, for \( f \) to be continuous on \((-\infty, \infty)\), \( c = \frac{2}{3}. \)

43. (a) \( f(x) = \frac{x^4 - 1}{x - 1} = \frac{(x^2 + 1)(x^2 - 1)}{x - 1} = \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = (x^2 + 1)(x + 1) \) \([or x^3 + x^2 + x + 1]\)

for \( x \neq 1 \). The discontinuity is removable and \( g(x) = x^3 + x^2 + x + 1 \) agrees with \( f \) for \( x \neq 1 \) and is continuous on \( \mathbb{R} \).

(b) \( f(x) = \frac{x^3 - x^2 - 2x}{x - 2} = \frac{x(x^2 - x - 2)}{x - 2} = \frac{x(x - 2)(x + 1)}{x - 2} = x(x + 1) \) \([or x^2 + x]\) \( \quad \) for \( x \neq 2 \). The discontinuity is removable and \( g(x) = x^2 + x \) agrees with \( f \) for \( x \neq 2 \) and is continuous on \( \mathbb{R} \).

(c) \( \lim_{{x \to \pi^-}} f(x) = \lim_{{x \to \pi^-}} [\sin x] = \lim_{{x \to \pi^-}} 0 = 0 \) and \( \lim_{{x \to \pi^+}} f(x) = \lim_{{x \to \pi^+}} [\sin x] = \lim_{{x \to \pi^+}} (-1) = -1 \), so \( \lim_{{x \to \pi}} f(x) \) does not exist. The discontinuity at \( x = \pi \) is a jump discontinuity.

45. \( f(x) = x^2 + 10 \sin x \) is continuous on the interval \([31, 32]\), \( f(31) \approx 957 \), and \( f(32) \approx 1030 \). Since \( 957 < 1000 < 1030 \), there is a number \( c \) in \((31, 32)\) such that \( f(c) = 1000 \) by the Intermediate Value Theorem. \( \text{Note: There is also a number } c \text{ in } (-32, -31) \text{ such that } f(c) = 1000. \)

47. \( f(x) = x^4 + x - 3 \) is continuous on the interval \([1, 2]\), \( f(1) = -1 \), and \( f(2) = 15 \). Since \(-1 < 0 < 15\), there is a number \( c \) in \((1, 2)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( x^4 + x - 3 = 0 \) in the interval \((1, 2)\).

49. \( f(x) = \cos x - x \) is continuous on the interval \([0, 1]\), \( f(0) = 1 \), and \( f(1) = \cos 1 - 1 \approx -0.46 \). Since \(-0.46 < 0 < 1\), there is a number \( c \) in \((0, 1)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( \cos x - x = 0 \), or \( \cos x = x \), in the interval \((0, 1)\).
51. (a) \( f(x) = \cos x - x^3 \) is continuous on the interval \([0, 1]\), \( f(0) = 1 > 0 \), and \( f(1) = \cos 1 - 1 \approx -0.46 < 0 \). Since 

\[ 1 > 0 > -0.46, \]

there is a number \( c \) in \((0, 1)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( \cos x - x^3 = 0 \), or \( \cos x = x^3 \), in the interval \((0, 1)\).

(b) \( f(0.86) \approx 0.016 > 0 \) and \( f(0.87) \approx -0.014 < 0 \), so there is a root between 0.86 and 0.87, that is, in the interval \((0.86, 0.87)\).

53. (a) Let \( f(x) = 100e^{-x/100} - 0.01x^2 \). Then \( f(0) = 100 > 0 \) and 

\[ f(100) = 100e^{-1} - 100 \approx -63.2 < 0. \]

So by the Intermediate Value Theorem, there is a number \( c \) in \((0, 100)\) such that \( f(c) = 0 \). 

This implies that \( 100e^{-c/100} = 0.01c^2 \).

(b) Using the intersect feature of the graphing device, we find that the root of the equation is \( x = 70.347 \), correct to three decimal places.

55. (\( \Rightarrow \)) If \( f \) is continuous at \( a \), then by Theorem 8 with \( g(h) = a + h \), we have

\[ \lim_{h \to 0} f(a + h) = f \left( \lim_{h \to 0} (a + h) \right) = f(a). \]

(\( \Leftarrow \)) Let \( \varepsilon > 0 \). Since \( \lim_{h \to 0} f(a + h) = f(a) \), there exists \( \delta > 0 \) such that \( 0 < |h| < \delta \implies |f(a + h) - f(a)| < \varepsilon \). So if \( 0 < |x - a| < \delta \), then \( |f(x) - f(a)| = |f(a + (x - a)) - f(a)| < \varepsilon \).

Thus, \( \lim_{x \to a} f(x) = f(a) \) and so \( f \) is continuous at \( a \).

57. As in the previous exercise, we must show that \( \lim_{h \to 0} \cos(a + h) = \cos a \) to prove that the cosine function is continuous.

\[ \lim_{h \to 0} \cos(a + h) = \lim_{h \to 0} \left( \cos a \cos h - \sin a \sin h \right) = \left( \lim_{h \to 0} \cos a \right) \left( \lim_{h \to 0} \cos h \right) - \left( \lim_{h \to 0} \sin a \right) \left( \lim_{h \to 0} \sin h \right) = (\cos a)(1) - (\sin a)(0) = \cos a \]

59. \( f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases} \) is continuous nowhere. For, given any number \( a \) and any \( \delta > 0 \), the interval \((a - \delta, a + \delta)\) contains both infinitely many rational and infinitely many irrational numbers. Since \( f(a) = 0 \) or 1, there are infinitely many numbers \( x \) with \( 0 < |x - a| < \delta \) and \( |f(x) - f(a)| = 1 \). Thus, \( \lim_{x \to a} f(x) \neq f(a) \). [In fact, \( \lim_{x \to a} f(x) \) does not even exist.]

61. If there is such a number, it satisfies the equation \( x^3 + 1 = x \implies x^3 - x + 1 = 0 \). Let the left-hand side of this equation be called \( f(x) \). Now \( f(-2) = -5 < 0 \), and \( f(-1) = 1 > 0 \). Note also that \( f(x) \) is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number \( c \) between -2 and -1 such that \( f(c) = 0 \), so that \( c = c^3 + 1 \).

63. \( f(x) = x^4 \sin(1/x) \) is continuous on \((-\infty, 0) \cup (0, \infty)\) since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since \(-1 \leq \sin(1/x) \leq 1 \), we have \(-x^4 \leq x^4 \sin(1/x) \leq x^4 \). Because
2.6 Limits at Infinity; Horizontal Asymptotes

1. (a) As \( x \) becomes large, the values of \( f(x) \) approach 5.

(b) As \( x \) becomes large negative, the values of \( f(x) \) approach 3.

3. (a) \( \lim_{x \to -2} f(x) = \infty \) \hspace{1cm} (b) \( \lim_{x \to -1^-} f(x) = \infty \) \hspace{1cm} (c) \( \lim_{x \to -1^+} f(x) = -\infty \)

(d) \( \lim_{x \to \infty} f(x) = 1 \) \hspace{1cm} (e) \( \lim_{x \to 0} f(x) = 2 \) \hspace{1cm} (f) Vertical: \( x = -1, x = 2 \); Horizontal: \( y = 1, y = 2 \)

5. \( f(0) = 0, \quad f(1) = 1, \quad f(x) = 0, \) \hspace{1cm} \( \lim_{x \to -2} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = \infty, \) \hspace{1cm} \( f(x) = 0, \quad \lim_{x \to 0^-} f(x) = \infty, \quad \lim_{x \to 0^+} f(x) = -\infty, \)

\( f(x) = 3, \quad \lim_{x \to -4^+} f(x) = -\infty, \quad \lim_{x \to -4^-} f(x) = -\infty, \quad \lim_{x \to -4} f(x) = \infty, \quad \lim_{x \to 4} f(x) = 3 \)

7. \( \lim_{x \to -\infty} f(x) = 0, \quad \lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to 0^-} f(x) = \infty, \quad \lim_{x \to 0^+} f(x) = -\infty, \)

11. If \( f(x) = x^2/2^x \), then a calculator gives \( f(0) = 0, f(1) = 0.5, f(2) = 1, f(3) = 1.125, f(4) = 1, f(5) = 0.78125, f(6) = 0.5625, f(7) = 0.3828125, f(8) = 0.25, f(9) = 0.158203125, f(10) = 0.09765625, f(20) \approx 0.00038147, f(50) \approx 2.2204 \times 10^{-12}, f(100) \approx 7.8886 \times 10^{-27}. \)

It appears that \( \lim_{x \to -\infty} (x^2/2^x) = 0. \)
13. \( \lim_{x \to \infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8} = \lim_{x \to \infty} \frac{(3x^2 - x + 4)/x^2}{(2x^2 + 5x - 8)/x^2} \) [divide both the numerator and denominator by \( x^2 \) (the highest power of \( x \) that appears in the denominator)]

= \( \lim_{x \to \infty} \frac{3 - 1/x + 4/x^2}{2 + 5/x - 8/x^2} \) [Limit Law]

= \( \lim_{x \to \infty} 3 - \lim_{x \to \infty} (1/x) + \lim_{x \to \infty} (4/x^2) \) [Limit Laws 1 and 2]

= \( \lim_{x \to \infty} 2 + \lim_{x \to \infty} (5/x) - \lim_{x \to \infty} (8/x^2) \)

= \( \frac{3}{2} + 5 \lim_{x \to \infty} (1/x) - 8 \lim_{x \to \infty} (1/x^2) \) [Limit Laws 7 and 3]

= \( 3 - 0 + 4(0) \)

= \( 2/5 \) (0) - 8(0)

= \( 3 \)

15. \( \lim_{x \to \infty} \frac{1}{2x + 3} = \lim_{x \to \infty} \frac{1/x}{(2x + 3)/x} = \lim_{x \to \infty} \frac{1/x}{(2 + 3/x)} = \lim_{x \to \infty} \frac{1/x}{2 + 3 \lim_{x \to \infty} (1/x)} = \frac{0}{2 + 3(0)} = \frac{0}{2} = 0 \)

17. \( \lim_{x \to \infty} \frac{1 - x - x^2}{2x^2 - 7} = \lim_{x \to \infty} \frac{(1 - x - x^2)/x^2}{(2x^2 - 7)/x^2} = \frac{\lim_{x \to \infty} (1/x^2) - 1/x - 1}{\lim_{x \to \infty} (2 - 7/x^2)} \)

= \( \frac{\lim_{x \to \infty} (1/x^2) - \lim_{x \to \infty} (1/x) - \lim_{x \to \infty} 1}{\lim_{x \to \infty} 2 - 7 \lim_{x \to \infty} (1/x^2)} = \frac{0 - 1}{2 - 7(0)} = \frac{1}{2} \)

19. Divide both the numerator and denominator by \( x^3 \) (the highest power of \( x \) that occurs in the denominator).

= \( \lim_{x \to \infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4} = \lim_{x \to \infty} \frac{x^3 + 5x}{x^3} - \frac{x^3 - x^2 + 4}{x^3} = \lim_{x \to \infty} \frac{1 + 5}{x^2} - \frac{1}{x^3} = \lim_{x \to \infty} \frac{1}{x^2} + \frac{4}{x^3} = \frac{\lim_{x \to \infty} \left(1 + \frac{5}{x^2}\right)}{\lim_{x \to \infty} \left(2 - \frac{1}{x} + \frac{4}{x^3}\right)} = \frac{1 + 5(0)}{2 - 0 + 4(0)} = \frac{1}{2} \)

21. First, multiply the factors in the denominator. Then divide both the numerator and denominator by \( u^4 \).

= \( \lim_{u \to \infty} \frac{4u^4 + 5}{(u^2 - 2)(2u^2 - 1)} = \lim_{u \to \infty} \frac{4u^4 + 5}{2u^4 - 5u^2 + 2} = \lim_{u \to \infty} \frac{4u^4 + 5}{u^4} - \frac{1}{2 - 5u^2 + 2} = \frac{\lim_{u \to \infty} \left(4 + \frac{5}{u^4}\right)}{\lim_{u \to \infty} \left(2 - \frac{5}{u^2} + \frac{2}{u^4}\right)} = \frac{4 + 5(0)}{2 - 5(0) + 2(0)} = \frac{4}{2} = 2 \)
23. \[ \lim_{x \to \infty} \frac{\sqrt[3]{x^5 - x}}{x^3 + 1} = \lim_{x \to \infty} \frac{\sqrt[3]{x^5 - x}}{(x^3 + 1)x^3} = \lim_{x \to \infty} \frac{\sqrt[3]{(9x^6 - x)/x^6}}{(1 + 1/x^3)} \] [since \( x^3 = \sqrt[3]{x^6} \) for \( x > 0 \)] 
\[ = \frac{\lim_{x \to \infty} \sqrt[3]{9 - 1/x^5}}{\lim_{x \to \infty} (1 + 1/x^3)} = \frac{\sqrt[3]{9 - \lim (1/x^5)}}{1 + 0} = \sqrt[3]{9 - 0} = 3 \]

25. \[ \lim_{x \to \infty} (\sqrt{x^2 + x} - 3x) = \lim_{x \to \infty} \frac{(\sqrt{x^2 + x} - 3x)(\sqrt{x^2 + x} + 3x)}{\sqrt{x^2 + x} + 3x} = \lim_{x \to \infty} \frac{9x^2 + x - 3x}{\sqrt{x^2 + x} + 3x} = \lim_{x \to \infty} \frac{x}{\frac{1}{\sqrt{x^2 + x} + 3x}} = \lim_{x \to \infty} \frac{1}{\frac{1}{9 + 1/x^2 + 3x}} = \frac{1}{3 + 3} = \frac{1}{6} \]

27. \[ \lim_{x \to \infty} (\sqrt{x^2 + ax} - \sqrt{x^2 + bx}) = \lim_{x \to \infty} \frac{(\sqrt{x^2 + ax} - \sqrt{x^2 + bx})(\sqrt{x^2 + ax} + \sqrt{x^2 + bx})}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} = \lim_{x \to \infty} \frac{(x^2 + ax) - (x^2 + bx)}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} = \lim_{x \to \infty} \frac{a - b}{\sqrt{a/x^2} + \sqrt{b/x^2}} = \frac{a - b}{a/b} = \frac{a - b}{2} \]

29. \[ \lim_{x \to \infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4} = \lim_{x \to \infty} \frac{(x + x^3 + x^5)/x^4}{(1 - x^2 + x^4)/x^4} = \lim_{x \to \infty} \frac{1}{1/x^3 + 1/x + x} = \infty \] [divide by the highest power of \( x \) in the denominator]

because \( (1/x^3 + 1/x + x) \to \infty \) and \( (1/x^4 - 1/x^2 + 1) \to 1 \) as \( x \to \infty \).

31. \[ \lim_{x \to -\infty} (x^4 + x^5) = \lim_{x \to -\infty} x^5(\frac{1}{x} + 1) = -\infty \] because \( x^5 \to -\infty \) and \( 1/x + 1 \to 1 \) as \( x \to -\infty \).

Or: \[ \lim_{x \to -\infty} (x^4 + x^5) = \lim_{x \to -\infty} x^4(1 + x) = -\infty. \]

33. \[ \lim_{x \to 1} \frac{1 - e^x}{1 + 2e^x} = \lim_{x \to 1} \frac{(1 - e^x)/e^x}{(1 + 2e^x)/e^x} = \lim_{x \to 1} \frac{1/e^x - 1}{0 + 2} = \frac{0 - 1}{2} = -\frac{1}{2} \]

35. Since \( -1 \leq \cos x \leq 1 \) and \( e^{-2x} > 0 \), we have \( -e^{-2x} \leq e^{-2x} \cos x \leq e^{-2x} \). We know that \( \lim_{x \to \infty} (-e^{-2x}) = 0 \) and \( \lim_{x \to \infty} (e^{-2x}) = 0 \), so by the Squeeze Theorem, \( \lim_{x \to \infty} (e^{-2x} \cos x) = 0. \)

37. (a) \[ \begin{array}{|c|c|c|} 
\hline 
\text{\( x \)} & \text{\( f(x) \)} \\
\hline 
-100 & -0.4999625 \\
-100,000 & -0.499962 \\
-1,000,000 & -0.4999996 \\
\hline 
\end{array} \]

From the graph of \( f(x) = \sqrt{x^2 + x + 1} + x \), we estimate the value of \( \lim_{x \to -\infty} f(x) \) to be \(-0.5\).
(c) \( \lim_{x \to -\infty} \sqrt{x^2 + x + 1 + x} = \lim_{x \to -\infty} \left( \frac{\sqrt{x^2 + x + 1} + x}{\sqrt{x^2 + x + 1} - x} \right) \left( \frac{\sqrt{x^2 + x + 1} - x}{\sqrt{x^2 + x + 1} - x} \right) = \lim_{x \to -\infty} \frac{(x + 1)/x}{(\sqrt{x^2 + x + 1} - x)/x} = \lim_{x \to -\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1} = \frac{1 + 0}{-\sqrt{1+0+0} - 1} = -\frac{1}{2} \)

Note that for \( x < 0 \), we have \( \sqrt{x^2} = |x| = -x \), so when we divide the radical by \( x \), with \( x < 0 \), we get

\[
\frac{1}{x} \sqrt{x^2 + x + 1} = -\frac{1}{\sqrt{x}} \sqrt{x^2 + x + 1} = -\sqrt{1 + (1/x) + (1/x^2)}.
\]

39. \( \lim_{x \to -\infty} \frac{2x + 1}{x - 2} = \lim_{x \to -\infty} \frac{2x + 1}{x - 2} = \lim_{x \to -\infty} \frac{2 + \frac{1}{x}}{1 - \frac{2}{x}} = \lim_{x \to -\infty} \frac{2 + \frac{1}{x}}{1 - \frac{2}{x}} = \frac{2 + 0}{1 - 0} = 2 \), so \( y = 2 \) is a horizontal asymptote.

The denominator \( x - 2 \) is zero when \( x = 2 \) and the numerator is not zero, so we investigate \( y = f(x) = \frac{2x + 1}{x - 2} \) as \( x \) approaches 2. \( \lim_{x \to 2^-} f(x) = -\infty \) because as \( x \to 2^- \) the numerator is positive and the denominator approaches 0 through negative values. Similarly, \( \lim_{x \to 2^+} f(x) = \infty \). Thus, \( x = 2 \) is a vertical asymptote.

The graph confirms our work.

41. \( \lim_{x \to -\infty} \frac{2x^2 + x - 1}{x^2 + x - 2} = \lim_{x \to -\infty} \frac{2x^2 + x - 1}{x^2} = \lim_{x \to -\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{2}{x^2}} = \lim_{x \to -\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 - \frac{2}{x^2}} = \frac{2 + 0 - 0}{1 + 0 - 2(0)} = 2 \), so \( y = 2 \) is a horizontal asymptote.

\[
y = f(x) = \frac{2x^2 + x - 1}{x^2 + x - 2} = \frac{(2x - 1)(x + 1)}{(x + 2)(x - 1)}, \text{ so } \lim_{x \to -2^-} f(x) = \infty, \lim_{x \to -2^+} f(x) = -\infty, \text{ and } \lim_{x \to -1^+} f(x) = \infty. \text{ Thus, } x = -2 \text{ and } x = 1 \text{ are vertical asymptotes. The graph confirms our work.}
\]

43. \( y = f(x) = \frac{x^3 - x}{x^2 - 6x + 5} = \frac{x(x^2 - 1)}{(x - 1)(x - 5)} = \frac{x(x+1)(x-1)}{(x-1)(x-5)} = \frac{x(x+1)}{x-5} = g(x) \text{ for } x \neq 1. \)

The graph of \( g \) is the same as the graph of \( f \) with the exception of a hole in the graph of \( f \) at \( x = 1 \). By long division, \( g(x) = \frac{x^2 + x}{x - 5} = x + 6 + \frac{30}{x - 5} \).

As \( x \to \pm \infty \), \( g(x) \to \pm \infty \), so there is no horizontal asymptote. The denominator of \( g \) is zero when \( x = 5 \). \( \lim_{x \to 5^-} g(x) = -\infty \) and \( \lim_{x \to 5^+} g(x) = \infty \), so \( x = 5 \) is a vertical asymptote. The graph confirms our work.
45. From the graph, it appears \( y = 1 \) is a horizontal asymptote.

\[
\lim_{x \to -\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} = \lim_{x \to -\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} = \lim_{x \to -\infty} \frac{3 + (500/x)}{1 + (500/x) + (100/x^2) + (2000/x^3)} = \frac{3 + 0}{1 + 0 + 0 + 0} = 3, \quad \text{so } y = 3 \text{ is a horizontal asymptote.}
\]

The discrepancy can be explained by the choice of the viewing window. Try \([-100,000, 100,000]\) by \([-1, 4]\) to get a graph that lends credibility to our calculation that \( y = 3 \) is a horizontal asymptote.

47. Let’s look for a rational function.

1. \( \lim_{x \to -\infty} f(x) = 0 \Rightarrow \text{degree of numerator} < \text{degree of denominator} \)

2. \( \lim_{x \to 0} f(x) = -\infty \Rightarrow \text{there is a factor of } x^2 \text{ in the denominator (not just } x, \text{ since that would produce a sign change at } x = 0, \text{ and the function is negative near } x = 0. \)

3. \( \lim_{x \to \infty} f(x) = \infty \text{ and } \lim_{x \to -\infty} f(x) = -\infty \Rightarrow \text{vertical asymptote at } x = 3; \text{ there is a factor of } (x - 3) \text{ in the denominator.} \)

4. \( f(2) = 0 \Rightarrow 2 \text{ is an } x\text{-intercept; there is at least one factor of } (x - 2) \text{ in the numerator.} \)

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us 

\[ f(x) = \frac{2 - x}{x^2(x - 3)} \text{ as one possibility.} \]

49. \( y = f(x) = x^4 - x^6 = x^4(1 - x^2) = x^4(1 + x)(1 - x). \) The \( y\)-intercept is \( f(0) = 0. \) The \( x\)-intercepts are 0, -1, and 1 [found by solving \( f(x) = 0 \) for \( x \)].

Since \( x^4 > 0 \) for \( x \neq 0, f \) doesn’t change sign at \( x = 0. \) The function does change sign at \( x = -1 \) and \( x = 1. \) As \( x \to \pm \infty, f(x) = x^4(1 - x^2) \) approaches \(-\infty\) because \( x^4 \to \infty \) and \( (1 - x^2) \to -\infty. \)

51. \( y = f(x) = (3 - x)(1 + x)^2(1 - x)^4. \) The \( y\)-intercept is \( f(0) = 3(1)^2(1)^4 = 3. \) The \( x\)-intercepts are 3, -1, and 1. There is a sign change at 3, but not at -1 and 1.

When \( x \) is large positive, \( 3 - x \) is negative and the other factors are positive, so

\[ \lim_{x \to -\infty} f(x) = -\infty. \]  When \( x \) is large negative, \( 3 - x \) is positive, so

\[ \lim_{x \to -\infty} f(x) = \infty. \]

53. (a) Since \( -1 \leq \sin x \leq 1 \) for all \( x, \) \( -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \) for \( x > 0. \) As \( x \to \infty, -1/x \to 0 \) and \( 1/x \to 0, \) so by the Squeeze Theorem, \( (\sin x)/x \to 0. \) Thus, \( \lim_{x \to \infty} \frac{\sin x}{x} = 0. \)
(b) From part (a), the horizontal asymptote is \( y = 0 \). The function
\[
y = \frac{\sin x}{x}
\]
crosses the horizontal asymptote whenever \( \sin x = 0 \);
that is, at \( x = \pi n \) for every integer \( n \). Thus, the graph crosses the
asymptote an infinite number of times.

55. Divide the numerator and the denominator by the highest power of \( x \) in \( Q(x) \).

(a) If \( \deg P < \deg Q \), then the numerator \( \to 0 \) but the denominator doesn’t. So
\[
\lim_{x \to \infty} \frac{P(x)}{Q(x)} = 0.
\]
(b) If \( \deg P > \deg Q \), then the numerator \( \to \pm \infty \) but the denominator doesn’t, so
\[
\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \pm \infty
\]
(depending on the ratio of the leading coefficients of \( P \) and \( Q \)).

57. \[
\lim_{x \to \infty} \frac{5 \sqrt{x}}{\sqrt{x} - 1} \cdot \frac{1}{\sqrt{x}} = \lim_{x \to \infty} \frac{5}{\sqrt{1 - (1/x)}} = 5
\]
\[
\lim_{x \to \infty} \frac{10e^x - 21}{2e^x} \cdot \frac{1}{e^x} = \lim_{x \to \infty} \frac{10 - 21e^{-x}}{2} = 10/2 = 5.
\]
Since \( 10e^x - 21 < f(x) < 5 \sqrt{x} \),
we have \( \lim_{x \to \infty} f(x) = 5 \) by the Squeeze Theorem.

59. (a) \[
\lim_{t \to \infty} v(t) = \lim_{t \to \infty} v^* \left(1 - e^{-\gamma t/v^*}\right) = v^* (1 - 0) = v^*
\]
(b) We graph \( v(t) = 1 - e^{-9.8t} \) and \( v(t) = 0.99v^* \), or in this case,
\( v(t) = 0.99 \). Using an intersect feature or zooming in on the point of
intersection, we find that \( t \approx 0.47 \) s.

61. Let \( g(x) = \frac{3x^2 + 1}{2x^2 + x + 1} \) and \( f(x) = |g(x) - 1.5| \). Note that
\[
\lim_{x \to \infty} g(x) = \frac{3}{2} \quad \text{and} \quad \lim_{x \to \infty} f(x) = 0.
\]
We are interested in finding the
\( x \)-value at which \( f(x) < 0.05 \). From the graph, we find that \( x \approx 14.804 \),
so we choose \( N = 15 \) (or any larger number).

63. For \( \varepsilon = 0.5 \), we need to find \( N \) such that \[
\left| \frac{\sqrt{4x^2 + 1}}{x + 1} - (-2) \right| < 0.5
\]
\(-2.5 < \frac{\sqrt{4x^2 + 1}}{x + 1} < -1.5 \) whenever \( x \leq N \). We graph the three parts of this
inequality on the same screen, and see that the inequality holds for \( x \leq -6 \). So we
choose \( N = -6 \) (or any smaller number).

For \( \varepsilon = 0.1 \), we need \(-2.1 < \frac{\sqrt{4x^2 + 1}}{x + 1} < -1.9 \) whenever \( x \leq N \). From the
graph, it seems that this inequality holds for \( x \leq -22 \). So we choose \( N = -22 \)
(or any smaller number).
65. (a) \(1/x^2 < 0.0001 \iff x^2 > 1/0.0001 = 10000 \iff x > 100 \ (x > 0)\)

(b) If \(\varepsilon > 0\) is given, then \(1/x^2 < \varepsilon \iff x^2 > 1/\varepsilon\). Let \(N = 1/\sqrt{\varepsilon}\).

Then \(x > N \Rightarrow x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \frac{1}{x^2} - 0 = \frac{1}{x^2} < \varepsilon\), so \(\lim_{x \to \infty} \frac{1}{x^2} = 0\).

67. For \(x < 0\), \(|1/x - 0| = -1/x\). If \(\varepsilon > 0\) is given, then \(-1/x < \varepsilon \iff x < -1/\varepsilon\).

Take \(N = -1/\varepsilon\). Then \(x < N \Rightarrow x < -1/\varepsilon \Rightarrow \left|(1/x) - 0\right| = -1/x < \varepsilon\), so \(\lim_{x \to -\infty} (1/x) = 0\).

69. Given \(M > 0\), we need \(N > 0\) such that \(x > N \Rightarrow e^x > M\). Now \(e^x > M \iff x > \ln M\), so take \(N = \max(1, \ln M)\). (This ensures that \(N > 0\).) Then \(x > N = \max(1, \ln M) \Rightarrow e^x > \max(e, M) \geq M\), so \(\lim_{x \to \infty} e^x = \infty\).

71. Suppose that \(\lim_{x \to \infty} f(x) = L\). Then for every \(\varepsilon > 0\) there is a corresponding positive number \(N\) such that \(|f(x) - L| < \varepsilon\) whenever \(x > N\). If \(t = 1/x\), then \(x > N \iff 0 < 1/x < 1/N \iff 0 < t < 1/N\). Thus, for every \(\varepsilon > 0\) there is a corresponding \(\delta > 0\) (namely \(1/N\)) such that \(|f(1/t) - L| < \varepsilon\) whenever \(0 < t < \delta\). This proves that \(\lim_{t \to 0^+} f(1/t) = L = \lim_{x \to \infty} f(x)\).

Now suppose that \(\lim_{x \to -\infty} f(x) = L\). Then for every \(\varepsilon > 0\) there is a corresponding negative number \(N\) such that \(|f(x) - L| < \varepsilon\) whenever \(x < N\). If \(t = 1/x\), then \(x < N \iff 1/N < 1/x < 0 \iff 1/N < t < 0\). Thus, for every \(\varepsilon > 0\) there is a corresponding \(\delta > 0\) (namely \(-1/N\)) such that \(|f(1/t) - L| < \varepsilon\) whenever \(-\delta < t < 0\). This proves that \(\lim_{t \to 0^-} f(1/t) = L = \lim_{x \to -\infty} f(x)\).

2.7 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: \(m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(3)}{x - 3}\).

(b) This is the limit of the slope of the secant line \(PQ\) as \(Q\) approaches \(P\): \(m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}\).

3. (a) (i) Using Definition 1 with \(f(x) = 4x - x^2\) and \(P(1, 3)\),

\[
m = \lim_{x \to 1} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{4x - x^2 - 3}{x - 1} = \lim_{x \to 1} \frac{-x^2 + 4x - 3}{x - 1} = \lim_{x \to 1} \frac{-(x - 1)(x - 3)}{x - 1} = \lim_{x \to 1} (3 - x) = 3 - 1 = 2
\]

(ii) Using Equation 2 with \(f(x) = 4x - x^2\) and \(P(1, 3)\),

\[
m = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{[4(1 + h) - (1 + h)^2] - 3}{h} = \lim_{h \to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} \leq \lim_{h \to 0} \frac{-h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(-h + 2)}{h} = \lim_{h \to 0} (-h + 2) = 2
\]

(b) An equation of the tangent line is \(y - f(a) = f'(a)(x - a) \Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - 3 = 2(x - 1)\), or \(y = 2x + 1\).
The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point (1, 3). Now zoom in toward the point (1, 3) until the parabola and the tangent line are indistinguishable.

5. Using (1) with $f(x) = \frac{x - 1}{x - 2}$ and $P(3, 2)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{x - 1}{x - 3} = \lim_{x \to 3} \frac{x - 2 - 2(x - 2)}{x - 3}$$

Tangent line: $y - 2 = -1(x - 3) \iff y = -x + 5$

7. Using (1), $m = \lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} 1 = 1$

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \iff y = \frac{1}{2}x + \frac{1}{2}$

9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{3 + 4(a + h)^2 - 2(a + h)^3 - (3 + 4a^2 - 2a^3)}{h}$$

(b) At (1, 5): $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line

is $y - 5 = 2(x - 1) \iff y = 2x + 3$.

At (2, 3): $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent

line is $y - 3 = -8(x - 2) \iff y = -8x + 19$.

11. (a) The particle is moving to the right when $s$ is increasing; that is, on the intervals (0, 1) and (4, 6). The particle is moving to the left when $s$ is decreasing; that is, on the interval (2, 3). The particle is standing still when $s$ is constant; that is, on the intervals (1, 2) and (3, 4).
(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval \((0, 1)\), the slope is \(\frac{3 - 0}{1 - 0} = 3\). On the interval \((2, 3)\), the slope is \(\frac{1 - 3}{3 - 2} = -2\). On the interval \((4, 6)\), the slope is \(\frac{3 - 1}{6 - 4} = 1\).

13. Let \(s(t) = 40t - 16t^2\).

\[
v(2) = \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t \to 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t \to 2} \frac{-8(2t^2 - 5t + 2)}{t - 2}
\]

\[
= \lim_{t \to 2} \frac{-8(t - 2)(2t - 1)}{t - 2} = -8 \lim_{t \to 2} (2t - 1) = -8(3) = -24
\]

Thus, the instantaneous velocity when \(t = 2\) is \(-24\) ft/s.

15. \(v(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h} = \lim_{h \to 0} \frac{1}{(a + h)^2} - \frac{1}{a^2} = \lim_{h \to 0} \frac{a^2 - (a + h)^2}{a^2(a + h)^2}
\]

\[
= \lim_{h \to 0} \frac{-2ah + h^2}{ha^2(a + h)^2} = \lim_{h \to 0} \frac{-h(2a + h)}{ha^2(a + h)^2} = \lim_{h \to 0} \frac{-2a + h}{a^2(a + h)^2} = -\frac{2a}{a^2} = -\frac{2}{a^2} \text{ m/s}
\]

So \(v(1) = -\frac{2}{1^3} = -2 \text{ m/s}, v(2) = -\frac{2}{2^3} = -\frac{1}{4} \text{ m/s}, \) and \(v(3) = -\frac{2}{3^3} = -\frac{2}{27} \text{ m/s}.

17. \(g'(0)\) is the only negative value. The slope at \(x = 4\) is smaller than the slope at \(x = 2\) and both are smaller than the slope at \(x = -2\). Thus, \(g'(0) < 0 < g'(4) < g'(2) < g'(-2)\).

19. We begin by drawing a curve through the origin with a slope of 3 to satisfy \(f(0) = 0\) and \(f'(0) = 3\). Since \(f'(1) = 0\), we will round off our figure so that there is a horizontal tangent directly over \(x = 1\). Last, we make sure that the curve has a slope of \(-1\) as we pass over \(x = 2\). Two of the many possibilities are shown.

21. Using Definition 2 with \(f(x) = 3x^2 - 5x\) and the point \((2, 2)\), we have

\[
f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{3(2 + h)^2 - 5(2 + h) - 2}{h} = \lim_{h \to 0} \frac{(12 + 12h + 3h^2 - 10 - 5h) - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{3h^2 + 7h}{h} = \lim_{h \to 0} (3h + 7) = 7
\]

So an equation of the tangent line at \((2, 2)\) is \(y - 2 = 7(x - 2)\) or \(y = 7x - 12\).
23. (a) Using Definition 2 with \( F(x) = \frac{5x}{1 + x^2} \) and the point \((2, 2)\), we have

\[
F'(2) = \lim_{h \to 0} \frac{F(2 + h) - F(2)}{h} = \lim_{h \to 0} \frac{\frac{5(2 + h)}{1 + (2 + h)^2} - 2}{h} = \lim_{h \to 0} \frac{5h + 10}{h^2 + 4h + 5} = \lim_{h \to 0} \frac{-2h^2 - 3h}{h} = \lim_{h \to 0} \frac{-2h - 3}{h^2 + 4h + 5} = \frac{-3}{5}
\]

So an equation of the tangent line at \((2, 2)\) is \( y - 2 = -\frac{3}{5} (x - 2) \) or \( y = -\frac{3}{5} x + \frac{16}{5} \).

25. Use Definition 2 with \( f(x) = 3 - 2x + 4x^2 \).

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{[3 - 2(a + h) + 4(a + h)^2] - (3 - 2a + 4a^2)}{h} = \lim_{h \to 0} \frac{-2a + 8ah + 4h^2}{h} = \lim_{h \to 0} (-2 + 8a + 4h) = -2 + 8a
\]

27. Use Definition 2 with \( f(t) = \frac{(2t + 1)}{(t + 3)} \).

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{2(a + h) + 1}{(a + h) + 3} = \lim_{h \to 0} \frac{(2a + 2h + 1)(a + 3) - (2a + 1)(a + h + 3)}{(a + h + 3)(a + 3)} = \frac{5}{(a + 3)^2}
\]

29. Use Definition 2 with \( f(x) = \frac{1}{\sqrt{x^2 + 2}} \).

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{(a + h)^2 + 2}} - \frac{1}{\sqrt{a^2 + 2}}}{h} = \lim_{h \to 0} \frac{\sqrt{a^2 + 2} - \sqrt{a + h + 2}}{h \sqrt{a^2 + 2} + \sqrt{a + h + 2} \sqrt{a^2 + 2}} = \lim_{h \to 0} \frac{(a + 2) - (a + h + 2)}{h + 2 \sqrt{a + 2} \sqrt{a + h + 2}} = -\frac{1}{(\sqrt{a^2 + 2})^2 (2 \sqrt{a^2 + 2})} = -\frac{1}{2(a + 2)^{3/2}}
\]

Note that the answers to Exercises 31 – 36 are not unique.

31. By Definition 2, \( \lim_{h \to 0} \frac{(1 + h)^{10} - 1}{h} = f'(1) \), where \( f(x) = x^{10} \) and \( a = 1 \).

Or: By Definition 2, \( \lim_{h \to 0} \frac{(1 + h)^{10} - 1}{h} = f'(0) \), where \( f(x) = (1 + x)^{10} \) and \( a = 0 \).
33. By Equation 3, \( \lim_{x \to 5} \frac{2^x - 32}{x - 5} = f'(5) \), where \( f(x) = 2^x \) and \( a = 5 \).

35. By Definition 2, \( \lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h} = f'(\pi) \), where \( f(x) = \cos x \) and \( a = \pi \).

Or: By Definition 2, \( \lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h} = f'(0) \), where \( f(x) = \cos(\pi + x) \) and \( a = 0 \).

37. \( v(5) = f'(5) = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} = \lim_{h \to 0} \frac{[100 + 50(5 + h) - 4.9(5 + h)^2] - [100 + 50(5) - 4.9(5)^2]}{h}
\)
\[ = \lim_{h \to 0} \frac{(100 + 250 + 50h - 4.9h^2 - 49h - 122.5) - (100 + 250 - 122.5)}{h} = \lim_{h \to 0} \frac{-4.9h^2 + h}{h} \]
\[ = \lim_{h \to 0} \frac{h(-4.9h + 1)}{h} = \lim_{h \to 0} (-4.9h + 1) = 1 \text{ m/s} \]

The speed when \( t = 5 \) is \( |1| = 1 \text{ m/s} \).

39. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.

41. (a) \[ \text{[2000, 2002]: } \frac{P(2002) - P(2000)}{2002 - 2000} = \frac{77 - 55}{2} = \frac{22}{2} = 11 \text{ percent/year} \]

(ii) \[ \text{[2000, 2001]: } \frac{P(2001) - P(2000)}{2001 - 2000} = \frac{68 - 55}{1} = 13 \text{ percent/year} \]

(iii) \[ \text{[1999, 2000]: } \frac{P(2000) - P(1999)}{2000 - 1999} = \frac{55 - 39}{1} = 16 \text{ percent/year} \]

(b) Using the values from (ii) and (iii), we have \( \frac{13 + 16}{2} = 14.5 \text{ percent/year} \).

(c) Estimating \( A \) as (1999, 40) and \( B \) as (2001, 70), the slope at 2000 is \( \frac{70 - 40}{2001 - 1999} = \frac{30}{2} = 15 \text{ percent/year} \).

43. (a) \( i) \frac{\Delta C}{\Delta x} = \frac{C(105) - C(100)}{105 - 100} = \frac{6601.25 - 6500}{5} = $20.25/\text{unit.} \)

(ii) \( \frac{\Delta C}{\Delta x} = \frac{C(101) - C(100)}{101 - 100} = \frac{6520.05 - 6500}{1} = $20.05/\text{unit.} \)

(b) \( \frac{C(100 + h) - C(100)}{h} = \frac{[5000 + 10(100 + h) + 0.05(100 + h)^2] - 6500}{h} = \frac{20h + 0.05h^2}{h} \)
\[ = 20 + 0.05h, h \neq 0 \]

So the instantaneous rate of change is \( \lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \to 0} (20 + 0.05h) = $20/\text{unit.} \)

45. (a) \( f'(x) \) is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.
(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is $17/ounce. So the cost of producing the 800th (or 801st) ounce is about $17.

(c) In the short term, the values of \( f'(x) \) will decrease because more efficient use is made of start-up costs as \( x \) increases. But eventually \( f'(x) \) might increase due to large-scale operations.

47. \( T'(10) \) is the rate at which the temperature is changing at 10:00 AM. To estimate the value of \( T'(10) \), we will average the difference quotients obtained using the times \( t = 8 \) and \( t = 12 \). Let \( A = \frac{T(8) - T(10)}{8 - 10} = \frac{72 - 81}{-2} = 4.5 \) and \( B = \frac{T(12) - T(10)}{12 - 10} = \frac{88 - 81}{2} = 3.5 \). Then \( T'(10) \approx \lim_{t \to 10} \frac{T(t) - T(10)}{t - 10} \approx \frac{A + B}{2} = \frac{4.5 + 3.5}{2} = 4^\circ \text{F/h} \).

49. (a) \( S'(T) \) is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are \( (\text{mg/L})/^\circ \text{C} \).

(b) For \( T = 16^\circ \text{C} \), it appears that the tangent line to the curve goes through the points \((0, 14)\) and \((32, 6)\). So \( S'(16) \approx \frac{6 - 14}{32 - 0} = \frac{-8}{32} = -0.25 \) \( (\text{mg/L})/^\circ \text{C} \). This means that as the temperature increases past \( 16^\circ \text{C} \), the oxygen solubility is decreasing at a rate of \( 0.25 \) \( (\text{mg/L})/^\circ \text{C} \).

51. Since \( f(x) = x \sin(1/x) \) when \( x \neq 0 \) and \( f(0) = 0 \), we have
\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin(1/h).
\] This limit does not exist since \( \sin(1/h) \) takes the values \(-1\) and \(1\) on any interval containing \(0\). (Compare with Example 4 in Section 2.2.)

2.8 The Derivative as a Function

1. It appears that \( f \) is an odd function, so \( f' \) will be an even function—that is, \( f'(-a) = f'(a) \).

(a) \( f'(-3) \approx 1.5 \)
(b) \( f'(-2) \approx 1 \)
(c) \( f'(-1) \approx 0 \)
(d) \( f'(0) \approx -4 \)
(e) \( f'(1) \approx 0 \)
(f) \( f'(2) \approx 1 \)
(g) \( f'(3) \approx 1.5 \)

3. (a) \( y' \) = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

(b) \( y' \) = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c) \( y' \) = I, since the slopes of the tangents to graph (c) are negative for \( x < 0 \) and positive for \( x > 0 \), as are the function values of graph I.

(d) \( y' \) = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.
Hints for Exercises 4–11: First plot $x$-intercepts on the graph of $f'$ for any horizontal tangents on the graph of $f$. Look for any corners on the graph of $f$—there will be a discontinuity on the graph of $f'$. On any interval where $f$ has a tangent with positive (or negative) slope, the graph of $f'$ will be positive (or negative). If the graph of the function is linear, the graph of $f'$ will be a horizontal line.

5. 

7. 

9. 

11. 

13. It appears that there are horizontal tangents on the graph of $M$ for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of $t$ on the graph of $M'$. The derivative is negative for the years 1963 to 1971.

15. 

The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As $x$ decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$. 
17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'(\frac{1}{2}) = 1$, $f'(1) = 2$, and $f'(2) = 4$.
(b) By symmetry, $f'(-x) = -f'(x)$. So $f'(-\frac{1}{2}) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.
(c) It appears that $f'(x)$ is twice the value of $x$, so we guess that $f'(x) = 2x$.

(d) $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

\[= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} \frac{h(2x + h)}{h} = \lim_{h \to 0} (2x + h) = 2x\]

19. $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \left[\frac{\frac{1}{2}(x + h) - \frac{1}{2}x}{h} - \frac{(\frac{1}{2}x - \frac{1}{2})}{h}\right] = \lim_{h \to 0} \frac{\frac{1}{2}h + \frac{1}{2}h - \frac{1}{2}x + \frac{1}{2}x}{h}$

\[= \lim_{h \to 0} \frac{\frac{1}{2}h}{h} = \lim_{h \to 0} \frac{1}{2} = \frac{1}{2}\]

Domain of $f$ = domain of $f'$ = $\mathbb{R}$.

21. $f'(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \lim_{h \to 0} \frac{[5(t + h) - 9(t + h)^2] - (5t - 9t^2)}{h}$

\[= \lim_{h \to 0} \frac{5t + 5h - 9(t^2 + 2th + h^2) - 5t + 9t^2}{h} = \lim_{h \to 0} \frac{5t + 5h - 9t^2 - 18th - 9h^2 - 5t + 9t^2}{h}
\]

\[= \lim_{h \to 0} \frac{5h - 18th - 9h^2}{h} = \lim_{h \to 0} \frac{h(5 - 18t - 9h)}{h} = \lim_{h \to 0} (5 - 18t - 9h) = 5 - 18t\]

Domain of $f$ = domain of $f'$ = $\mathbb{R}$.

23. $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x + h)^3 - 3(x + h) + 5] - (x^3 - 3x + 5)}{h}$

\[= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h + 5) - (x^3 - 3x + 5)}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}
\]

\[= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3\]

Domain of $f$ = domain of $f'$ = $\mathbb{R}$.

25. $g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} \frac{\sqrt{1 + 2x + h} - \sqrt{1 + 2x}}{h} \left[\frac{\sqrt{1 + 2x + h} + \sqrt{1 + 2x}}{\sqrt{1 + 2x + h} + \sqrt{1 + 2x}}\right]$ 

\[= \lim_{h \to 0} \frac{\sqrt{1 + 2x + h} - (1 + 2x)}{h} = \lim_{h \to 0} \frac{2}{\sqrt{1 + 2x + h} + \sqrt{1 + 2x}} = \frac{2}{2\sqrt{1 + 2x}} = \frac{1}{\sqrt{1 + 2x}}
\]

Domain of $g = [-\frac{1}{2}, \infty)$, domain of $g' = (-\frac{1}{2}, \infty)$. 

\[\text{TX.10}
\]

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27. \( G'(t) = \lim_{h \to 0} \frac{G(t+h) - G(t)}{h} = \lim_{h \to 0} \frac{4(t+h) - 4t}{(t+h+1)(t+1)} = \lim_{h \to 0} \frac{4t + 4t + 4h - 4t}{h(t+h+1)(t+1)} = \lim_{h \to 0} \frac{4h}{h(t+h+1)(t+1)} = \frac{4}{(t+1)^2} \)

Domain of \( G = \) domain of \( G' = (-\infty, -1) \cup (-1, \infty) \).

29. \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \)

Domain of \( f = \) domain of \( f' = \mathbb{R} \).

31. (a) \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{((x+h)^4 + 2(x+h)) - (x^4 + 2x)}{h} = \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h} = \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h} = \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2 \)

(b) Notice that \( f'(x) = 0 \) when \( f \) has a horizontal tangent, \( f'(x) \) is positive when the tangents have positive slope, and \( f'(x) \) is negative when the tangents have negative slope.

33. (a) \( U'(t) \) is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find \( U'(t) \), we use \( \lim_{h \to 0} \frac{U(t+h) - U(t)}{h} \approx \frac{U(t+h) - U(t)}{h} \) for small values of \( h \).

For 1993: \( U'(1993) \approx \frac{U(1994) - U(1993)}{1994 - 1993} = \frac{6.1 - 6.9}{1} = -0.80 \)

For 1994: We estimate \( U'(1994) \) by using \( h = -1 \) and \( h = 1 \), and then average the two results to obtain a final estimate.

\( h = -1 \Rightarrow U'(1994) \approx \frac{U(1993) - U(1994)}{1993 - 1994} = \frac{6.9 - 6.1}{-1} = -0.80; \)

\( h = 1 \Rightarrow U'(1994) \approx \frac{U(1995) - U(1994)}{1995 - 1994} = \frac{5.6 - 6.1}{1} = -0.50. \)

So we estimate that \( U'(1994) \approx \frac{1}{2}([-0.80] + (-0.50)) = -0.65 \).

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35. \( f \) is not differentiable at \( x = -4 \), because the graph has a corner there, and at \( x = 0 \), because there is a discontinuity there.

37. \( f \) is not differentiable at \( x = -1 \), because the graph has a vertical tangent there, and at \( x = 4 \), because the graph has a corner there.

39. As we zoom in toward \((-1, 0)\), the curve appears more and more like a straight line, so \( f(x) = x + \sqrt{|x|} \) is differentiable at \( x = -1 \). But no matter how much we zoom in toward the origin, the curve doesn’t straighten out—we can’t eliminate the sharp point (a cusp). So \( f \) is not differentiable at \( x = 0 \).

41. \( a = f, b = f', c = f'' \). We can see this because where \( a \) has a horizontal tangent, \( b = 0 \), and where \( b \) has a horizontal tangent, \( c = 0 \). We can immediately see that \( c \) can be neither \( f \) nor \( f' \), since at the points where \( c \) has a horizontal tangent, neither \( a \) nor \( b \) is equal to 0.

43. We can immediately see that \( a \) is the graph of the acceleration function, since at the points where \( a \) has a horizontal tangent, neither \( c \) nor \( b \) is equal to 0. Next, we note that \( a = 0 \) at the point where \( b \) has a horizontal tangent, so \( b \) must be the graph of the velocity function, and hence, \( b' = a \). We conclude that \( c \) is the graph of the position function.

45. 
\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1 + 4(x + h) - (x + h)^2} {h} - \frac{(1 + 4x - x^2)}{h} \\
&= \lim_{h \to 0} \frac{(1 + 4x + 4h - x^2 - 2xh - h^2) - (1 + 4x - x^2)}{h} = \lim_{h \to 0} \frac{4h - 2xh - h^2}{h} = \lim_{h \to 0} (4 - 2x - h) = 4 - 2x
\end{align*}
\]

\[
\begin{align*}
f''(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{4 - 2(x + h)}{h} - \frac{(4 - 2x)}{h} = \lim_{h \to 0} \frac{-2h}{h} = \lim_{h \to 0} (-2) = -2
\end{align*}
\]

We see from the graph that our answers are reasonable because the graph of \( f' \) is that of a linear function and the graph of \( f'' \) is that of a constant function.
47. 
\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[2(x+h)^2 - (x+h)^3] - (2x^2 - x^3)}{h} \]
\[ = \lim_{h \to 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h \to 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2 \]

\[ f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[4(x+h) - 3(x+h)^2] - (4x - 3x^2)}{h} = \lim_{h \to 0} \frac{h(4 - 6x - 3h)}{h} = 4 - 6x \]

\[ f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \to 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h \to 0} \frac{-6h}{h} = \lim_{h \to 0} (-6) = -6 \]

\[ f^{(4)}(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h \to 0} \frac{-6 - (-6)}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} (0) = 0 \]

The graphs are consistent with the geometric interpretations of the derivatives because \( f' \) has zeros where \( f \) has a local minimum and a local maximum, \( f'' \) has a zero where \( f' \) has a local maximum, and \( f''' \) is a constant function equal to the slope of \( f'' \).

49. (a) Note that we have factored \( x - a \) as the difference of two cubes in the third step.

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})} \]
\[ = \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \frac{1}{3a^{-2/3}} \]

(b) \( f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} \). This function increases without bound, so the limit does not exist, and therefore \( f'(0) \) does not exist.

(c) \( \lim_{x \to 0} |f'(x)| = \lim_{x \to 0} \frac{1}{3x^{2/3}} = \infty \) and \( f \) is continuous at \( x = 0 \) (root function), so \( f \) has a vertical tangent at \( x = 0 \).

51. \( f(x) = |x-6| = \begin{cases} 
  x - 6 & \text{if } x - 6 \geq 0 \\
  -(x - 6) & \text{if } x - 6 < 0 
\end{cases} \]
\[ = \begin{cases} 
  x - 6 & \text{if } x \geq 6 \\
  6 - x & \text{if } x < 6 
\end{cases} \]

So the right-hand limit is \( \lim_{x \to 6^+} \frac{f(x) - f(6)}{x - 6} = \lim_{x \to 6^+} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6^+} \frac{x - 6}{x - 6} = \lim_{x \to 6^+} 1 = 1 \), and the left-hand limit
\[ \lim_{x \to 6^-} \frac{f(x) - f(6)}{x - 6} = \lim_{x \to 6^-} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6^-} \frac{6 - x}{x - 6} = \lim_{x \to 6^-} (-1) = -1 \]
Since these limits are not equal,
\[ f'(6) = \lim_{x \to 6} \frac{f(x) - f(6)}{x - 6} \text{ does not exist and } f \text{ is not differentiable at } 6. \]

However, a formula for \( f' \) is \( f'(x) = \begin{cases} 
  1 & \text{if } x > 6 \\
  -1 & \text{if } x < 6 
\end{cases} \)

Another way of writing the formula is \( f'(x) = \frac{x - 6}{|x - 6|} \).
53. (a) \( f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \)

(b) Since \( f(x) = x^2 \) for \( x \geq 0 \), we have \( f'(x) = 2x \) for \( x > 0 \).

[See Exercise 2.8.17(d).] Similarly, since \( f(x) = -x^2 \) for \( x < 0 \), we have \( f'(x) = -2x \) for \( x < 0 \). At \( x = 0 \), we have

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x|x|}{x} = \lim_{x \to 0} |x| = 0.
\]

So \( f \) is differentiable at 0. Thus, \( f \) is differentiable for all \( x \).

(c) From part (b), we have \( f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} = 2|x| \).

55. (a) If \( f \) is even, then

\[
f'(-x) = \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x - h)] - f(-x)}{h}
= \lim_{h \to 0} \frac{f(x - h) - f(x)}{h} = -\lim_{h \to 0} \frac{f(x - h) - f(x)}{-h}
= -\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = -f'(x)
\]

Therefore, \( f' \) is odd.

(b) If \( f \) is odd, then

\[
f'(-x) = \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x - h)] - f(-x)}{h}
= \lim_{h \to 0} \frac{-f(x - h) + f(x)}{h} = \lim_{h \to 0} \frac{f(x - h) - f(x)}{-h}
= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)
\]

Therefore, \( f' \) is even.

57. In the right triangle in the diagram, let \( \Delta y \) be the side opposite \( \phi \) and \( \Delta x \) the side adjacent angle \( \phi \). Then the slope of the tangent line \( \ell \) is \( m = \Delta y/\Delta x = \tan \phi \). Note that \( 0 < \phi < \frac{\pi}{2} \). We know (see Exercise 17) that the derivative of \( f(x) = x^2 \) is \( f'(x) = 2x \). So the slope of the tangent to the curve at the point \((1, 1)\) is 2. Thus, \( \phi \) is the angle between 0 and \( \frac{\pi}{2} \), whose tangent is 2; that is, \( \phi = \tan^{-1} 2 \approx 63^\circ \).
2. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.

![Graphs showing different types of discontinuities](image)

The left- and right-hand limits are not equal.

There is an infinite discontinuity.

There are an infinite number of oscillations.

3. (a)–(g) See the statements of Limit Laws 1–6 and 11 in Section 2.3.

4. See Theorem 3 in Section 2.3.

5. (a) See Definition 2.2.6 and Figures 12–14 in Section 2.2.

(b) See Definition 2.6.3 and Figures 3 and 4 in Section 2.6.

6. (a) $y = x^2$: No asymptote

(b) $y = \sin x$: No asymptote

(c) $y = \tan x$: Vertical asymptotes $x = \frac{\pi}{2} + \pi n$, $n$ an integer

(d) $y = \tan^{-1} x$: Horizontal asymptotes $y = \pm \frac{\pi}{2}$

(e) $y = e^x$: Horizontal asymptote $y = 0$

(f) $y = \ln x$: Vertical asymptote $x = 0$

\[
\lim_{x \to -\infty} e^x = 0
\]

(g) $y = 1/x$: Vertical asymptote $x = 0$; horizontal asymptote $y = 0$

(h) $y = \sqrt{x}$: No asymptote

7. (a) A function $f$ is continuous at a number $a$ if $f(x)$ approaches $f(a)$ as $x$ approaches $a$; that is, $\lim_{x \to a} f(x) = f(a)$.

(b) A function $f$ is continuous on the interval $(-\infty, \infty)$ if $f$ is continuous at every real number $a$. The graph of such a function has no breaks and every vertical line crosses it.
8. See Theorem 2.5.10.
9. See Definition 2.7.1.
10. See the paragraph containing Formula 3 in Section 2.7.
11. (a) The average rate of change of \( y \) with respect to \( x \) over the interval \([x_1, x_2]\) is
    \[ \frac{f(x_2) - f(x_1)}{x_2 - x_1} \, . \]
    (b) The instantaneous rate of change of \( y \) with respect to \( x \) at \( x = x_1 \) is
    \[ \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \, . \]
12. See Definition 2.7.2. The pages following the definition discuss interpretations of \( \frac{f(x_2) - f(x_1)}{x_2 - x_1} \) as the slope of a tangent line to the graph of \( f \) at \( x = a \) and as an instantaneous rate of change of \( f(x) \) with respect to \( x \) when \( x = a \).
13. See the paragraphs before and after Example 6 in Section 2.8.
14. (a) A function \( f \) is differentiable at a number \( a \) if its derivative \( f' \) exists at \( x = a \); that is, if \( f'(a) \) exists.
    (c) See Theorem 2.8.4. This theorem also tells us that if \( f \) is not continuous at \( a \), then \( f \) is not differentiable at \( a \).
15. See the discussion and Figure 7 on page 159.

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don’t).
3. True. Limit Law 5 applies.
5. False. Consider \( \lim_{x \to 5} \frac{x(x - 5)}{x - 5} \) or \( \lim_{x \to 0} \frac{\sin(x - 5)}{x - 5} \). The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).
7. True. A polynomial is continuous everywhere, so \( \lim_{x \to b} p(x) \) exists and is equal to \( p(b) \).
9. True. See Figure 8 in Section 2.6.
11. False. Consider \( f(x) = \begin{cases} \frac{1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \).
13. True. Use Theorem 2.5.8 with \( a = 2, b = 5 \), and \( g(x) = 4x^2 - 11 \). Note that \( f(4) = 3 \) is not needed.
15. True, by the definition of a limit with \( \varepsilon = 1 \).
17. False. See the note after Theorem 4 in Section 2.8.
19. False. \( \frac{d^2 y}{dx^2} \) is the second derivative while \( \left( \frac{dy}{dx} \right)^2 \) is the first derivative squared. For example, if \( y = x \), then \( \frac{d^2 y}{dx^2} = 0 \), but \( \left( \frac{dy}{dx} \right)^2 = 1 \).
1. (a) \( \lim_{x \to 2^+} f(x) = 3 \) \( \lim_{x \to 3} f(x) \) does not exist since the left and right limits are not equal. (The left limit is \(-2\).)

(b) The equations of the horizontal asymptotes are \( y = -1 \) and \( y = 4 \).

(c) The equations of the vertical asymptotes are \( x = 0 \) and \( x = 2 \).

(d) \( f \) is discontinuous at \( x = -3, 0, 2, \) and \( 4 \). The discontinuities are jump, infinite, infinite, and removable, respectively.

3. Since the exponential function is continuous, \( \lim_{x \to 1} e^{3x-x} = e^{1-1} = e^0 = 1 \).

5. \( \lim_{x \to 3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \to 3} \frac{(x + 3)(x - 3)}{(x + 3)(x - 1)} = \lim_{x \to 3} \frac{x - 3}{x - 1} = -3 - 3 = -6 = \frac{3}{2} \)

7. \( \lim_{h \to 0} \frac{(h - 1)^3 + 1}{h} = \lim_{h \to 0} \frac{(h - 3)h^2 + 3h - 1}{h} = \lim_{h \to 0} \frac{h^3 - 3h^2 + 3h}{h} = \lim_{h \to 0} (h^2 - 3h + 3) = 3 \)

Another solution: Factor the numerator as a sum of two cubes and then simplify.

9. \( \lim_{r \to 9} \frac{\sqrt{r}}{(r - 9)^2} = \infty \) since \((r - 9)^4 \to 0\) as \( r \to 9 \) and \( \frac{\sqrt{r}}{(r - 9)^2} \to 0 \) for \( r \neq 9 \).

11. \( \lim_{u \to 1} \frac{u^4 - 1}{u^3 + 5u^2 - 6u} = \lim_{u \to 1} \frac{(u^2 + 1)(u^2 - 1)}{u(u^2 + 5u - 6)} = \lim_{u \to 1} \frac{(u^2 + 1)(u + 1)(u - 1)}{u(u + 6)(u - 1)} = \lim_{u \to 1} \frac{u^2 + 1}{u(u + 6)} = \frac{2(2)}{1(7)} = \frac{4}{7} \)

13. Since \( x \) is positive, \( \sqrt{x^2} = |x| = x \). Thus,

\[
\lim_{x \to \infty} \frac{\sqrt{x^2 - 9}}{2x - 6} = \lim_{x \to \infty} \frac{\sqrt{x^2 - 9}/\sqrt{x^2}}{(2x - 6)/x} \quad \lim_{x \to \infty} \frac{\sqrt{1 - 9/x^2}}{2 - 6/x} = \sqrt{1 - 0} = \frac{1}{2}
\]

15. Let \( t = \sin x \). Then as \( x \to \pi^- \), \( \sin x \to 0^+ \), so \( t \to 0^+ \). Thus, \( \lim_{x \to \pi^-} \ln(\sin x) = \lim_{t \to 0^+} \ln t = -\infty \).

17. \( \lim_{x \to \infty} \frac{\sqrt{x^2} + 4x + 1}{x - 1} = \lim_{x \to \infty} \left[ \frac{\sqrt{x^2} + 4x + 1 - x}{1} \right] = \lim_{x \to \infty} \left[ \frac{x^2 + 4x + 1}{\sqrt{x^2} + 4x + 1} \right] = \lim_{x \to \infty} \left[ \frac{(x^2 + 4x + 1) - x^2}{\sqrt{x^2} + 4x + 1} \right] = \lim_{x \to \infty} \left[ \frac{(4x + 1)x}{\sqrt{x^2} + 4x + 1} \right] = \lim_{x \to \infty} \left[ \frac{4 + 1/x}{\sqrt{1 + 4x + 1/x^2} + 1} \right] = \frac{4 + 0}{\sqrt{1 + 0 + 0 + 1}} = \frac{4}{2} = 2 \)
19. Let \( t = 1/x \). Then as \( x \to 0^+ \), \( t \to \infty \), and \( \lim_{x \to 0^+} \tan^{-1}(1/x) = \lim_{t \to \infty} \tan^{-1} t = \frac{\pi}{2} \).

21. From the graph of \( y = (\cos^2 x)/x^2 \), it appears that \( y = 0 \) is the horizontal asymptote and \( x = 0 \) is the vertical asymptote. Now \( 0 \leq (\cos x)^2 \leq 1 \) \( \Rightarrow \frac{0}{x^2} \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2} \). But \( \lim_{x \to \pm \infty} \frac{\cos^2 x}{x^2} = 0 \) and \( \lim_{x \to \pm \infty} \frac{\cos^2 x}{x^2} = 0 \), so by the Squeeze Theorem, \( \lim_{x \to \pm \infty} \frac{\cos^2 x}{x^2} = 0 \). Thus, \( y = 0 \) is the horizontal asymptote.

Thus, \( y = 0 \) is the horizontal asymptote, \( \lim_{x \to 0} \frac{\cos^2 x}{x^2} = \infty \) because \( \cos^2 x \to 1 \) and \( x^2 \to 0 \) as \( x \to 0 \), so \( x = 0 \) is the vertical asymptote.

23. Since \( 2x - 1 \leq f(x) \leq x^2 \) for \( 0 < x < 3 \) and \( \lim_{x \to 1} (2x - 1) = 1 = \lim_{x \to 1} x^2 \), we have \( \lim_{x \to 1} f(x) = 1 \) by the Squeeze Theorem.

25. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then \( |(14 - 5x) - 4| < \varepsilon \). But \( |(14 - 5x) - 4| < \varepsilon \) \( \iff \left| -5x + 10 \right| < \varepsilon \) \( \iff \left| -5 \right| |x - 2| < \varepsilon \) \( \iff |x - 2| < \varepsilon /5 \). So if we choose \( \delta = \varepsilon /5 \), then \( 0 < |x - 2| < \delta \) \( \Rightarrow \left| 14 - 5x \right| < \varepsilon \). Thus, \( \lim_{x \to 2} (14 - 5x) = 4 \) by the definition of a limit.

27. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then \( |x^2 - 3x - (-2)| < \varepsilon \). First, note that if \( |x - 2| < 1 \), then \( -1 < x - 2 < 1 \), so \( 0 < x - 1 < 2 \) \( \Rightarrow |x - 1| < 2 \). Now let \( \delta = \min \{ \varepsilon /2, 1 \} \). Then \( 0 < |x - 2| < \delta \) \( \Rightarrow |x^2 - 3x - (-2)| = |(x - 2)(x - 1)| < |x - 2||x - 1| < (\varepsilon /2)(2) = \varepsilon \).

Thus, \( \lim_{x \to 2} (x^2 - 3x) = -2 \) by the definition of a limit.

29. (a) \( f(x) = \sqrt{-x} \) if \( x < 0 \), \( f(x) = 3 - x \) if \( 0 \leq x < 3 \), \( f(x) = (x - 3)^2 \) if \( x > 3 \).

\( \begin{align*}
(i) & \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3 - x) = 3 \\
(ii) & \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0 \\
(iii) & \quad \text{Because of (i) and (ii), } \lim_{x \to 0} f(x) \text{ does not exist.} \\
(iv) & \quad \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3 - x) = 0 \\
(v) & \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3)^2 = 0 \\
(vi) & \quad \text{Because of (iv) and (v), } \lim_{x \to 3} f(x) = 0.
\end{align*} \)

(b) \( f \) is discontinuous at \( 0 \) since \( \lim_{x \to 0} f(x) \) does not exist.

\( f \) is discontinuous at \( 3 \) since \( f(3) \) does not exist.

31. \( \sin x \) is continuous on \( \mathbb{R} \) by Theorem 7 in Section 2.5. Since \( e^x \) is continuous on \( \mathbb{R} \), \( e^{\sin x} \) is continuous on \( \mathbb{R} \) by Theorem 9 in Section 2.5. Lastly, \( x \) is continuous on \( \mathbb{R} \) since it’s a polynomial and the product \( xe^{\sin x} \) is continuous on its domain \( \mathbb{R} \) by Theorem 4 in Section 2.5.

33. \( f(x) = 2x^3 + x^2 + 2 \) is a polynomial, so it is continuous on \([-2, -1] \) and \( f(-2) = -10 < 0 < 1 = f(-1) \). So by the Intermediate Value Theorem there is a number \( c \) in \((-2, -1)\) such that \( f(c) = 0 \), that is, the equation \( 2x^3 + x^2 + 2 = 0 \) has a root in \((-2, -1)\).
35. (a) The slope of the tangent line at (2, 1) is
\[
\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{9 - 2x^2 - 1}{x - 2} = \lim_{x \to 2} \frac{8 - 2x^2}{x - 2} = \lim_{x \to 2} \frac{-2(x^2 - 4)}{x - 2} = \lim_{x \to 2} \frac{-2(x - 2)(x + 2)}{x - 2}
\]
\[= \lim_{x \to 2} (-2(x + 2)) = -2 \cdot 4 = -8 \]
(b) An equation of this tangent line is \( y - 1 = -8(x - 2) \) or \( y = -8x + 17 \).

37. (a) \( s = s(t) = 1 + 2t + t^2/4 \). The average velocity over the time interval \([1, 1 + h]\) is
\[v_{\text{ave}} = \frac{s(1 + h) - s(1)}{(1 + h) - 1} = \frac{1 + 2(1 + h) + (1 + h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}\]
So for the following intervals the average velocities are:
(i) \([1, 3]\): \( h = 2 \), \( v_{\text{ave}} = (10 + 2)/4 = 3 \text{ m/s} \)
(ii) \([1, 2]\): \( h = 1 \), \( v_{\text{ave}} = (10 + 1)/4 = 2.75 \text{ m/s} \)
(iii) \([1, 1.5]\): \( h = 0.5 \), \( v_{\text{ave}} = (10 + 0.5)/4 = 2.625 \text{ m/s} \)
(iv) \([1, 1.1]\): \( h = 0.1 \), \( v_{\text{ave}} = (10 + 0.1)/4 = 2.525 \text{ m/s} \)
(b) When \( t = 1 \), the instantaneous velocity is \( \lim_{h \to 0} \frac{s(1 + h) - s(1)}{h} = \lim_{h \to 0} \frac{10h + h^2}{4h} = \frac{10}{4} = 2.5 \text{ m/s} \).

39. (a) \( f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^3 - 2x - 4}{x - 2} \)
\[= \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 2)}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 2) = 10 \]
(b) \( y - 4 = 10(x - 2) \) or \( y = 10x - 16 \)

41. (a) \( f'(r) \) is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).
(b) The total cost of paying off the loan is increasing by $1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately $1200.
(c) As \( r \) increases, \( C \) increases. So \( f'(r) \) will always be positive.
45. (a) \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{3 - 5(x + h)} - \sqrt{3 - 5x}}{h} = \lim_{h \to 0} \frac{\sqrt{3 - 5(x + h)} - \sqrt{3 - 5x}}{h} \frac{\sqrt{3 - 5(x + h)} + \sqrt{3 - 5x}}{\sqrt{3 - 5(x + h)} + \sqrt{3 - 5x}} \\
= \lim_{h \to 0} \frac{3 - 5(x + h) - (3 - 5x)}{h(\sqrt{3 - 5(x + h)} + \sqrt{3 - 5x})} = \lim_{h \to 0} \frac{-5}{\sqrt{3 - 5(x + h)} + \sqrt{3 - 5x}} = \frac{-5}{2\sqrt{3 - 5x}} \]

(b) Domain of \( f \): (the radicand must be nonnegative) \( 3 - 5x \geq 0 \) \( \Rightarrow \) \( 5x \leq 3 \) \( \Rightarrow \) \( x \in (-\infty, \frac{3}{5}] \)

Domain of \( f' \): exclude \( \frac{3}{5} \) because it makes the denominator zero;

\( x \in (-\infty, \frac{3}{5}) \)

(c) Our answer to part (a) is reasonable because \( f'(x) \) is always negative and \( f \) is always decreasing.

47. \( f \) is not differentiable: at \( x = -4 \) because \( f \) is not continuous, at \( x = -1 \) because \( f \) has a corner, at \( x = 2 \) because \( f \) is not continuous, and at \( x = 5 \) because \( f \) has a vertical tangent.

49. \( C'(1990) \) is the rate at which the total value of US currency in circulation is changing in billions of dollars per year. To estimate the value of \( C'(1990) \), we will average the difference quotients obtained using the times \( t = 1985 \) and \( t = 1995 \).

Let \( A = \frac{C(1985) - C(1990)}{1985 - 1990} = \frac{187.3 - 271.9}{-5} = \frac{-84.6}{-5} = 16.92 \) and

\( B = \frac{C(1995) - C(1990)}{1995 - 1990} = \frac{409.3 - 271.9}{5} = \frac{137.4}{5} = 27.48 \). Then

\( C'(1990) = \lim_{t \to 1990} \frac{C(t) - C(1990)}{t - 1990} \approx \frac{A + B}{2} = \frac{16.92 + 27.48}{2} = \frac{44.4}{2} = 22.2 \) billion dollars/year.

51. \( |f(x)| \leq g(x) \iff -g(x) \leq f(x) \leq g(x) \) and \( \lim_{x \to a} g(x) = 0 = \lim_{x \to a} -g(x) \).

Thus, by the Squeeze Theorem, \( \lim_{x \to a} f(x) = 0 \).
PROBLEMS PLUS

1. Let \( t = \sqrt[3]{x} \), so \( x = t^6 \). Then \( t \to 1 \) as \( x \to 1 \), so
\[
\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt[3]{x} - 1} = \lim_{t \to 1} \frac{t^2 - 1}{t^3 - 1} = \lim_{t \to 1} \frac{(t-1)(t+1)}{(t-1)(t^2 + t + 1)} = \lim_{t \to 1} \frac{t + 1}{t^2 + t + 1} = \frac{1 + 1}{1^2 + 1 + 1} = \frac{2}{3}.
\]

Another method: Multiply both the numerator and the denominator by \((\sqrt[3]{x} + 1)\). \(\sqrt[3]{x} + \sqrt[3]{x} + 1\).

3. For \(-\frac{1}{2} < x < \frac{1}{2}\), we have \(2x - 1 < 0\) and \(2x + 1 > 0\), so \(2x - 1 = -(2x - 1)\) and \(2x + 1 = 2x + 1\).

Therefore, \(\lim_{x \to 0} \frac{2x - 1}{x} - \frac{2x + 1}{x} = \lim_{x \to 0} \frac{-2x - 1 - (2x + 1)}{x} = \lim_{x \to 0} \frac{-4x}{x} = \lim_{x \to 0} (-4) = -4\).

5. Since \([x] \leq x < [x] + 1\), we have \(\frac{[x]}{[x]} \leq \frac{x}{[x]} < \frac{[x] + 1}{[x]} \Rightarrow 1 \leq \frac{x}{[x]} < 1 + \frac{1}{[x]} \) for \(x \geq 1\). As \(x \to \infty\), \([x] \to \infty\), so \(\frac{1}{[x]} \to 0\) and \(1 + \frac{1}{[x]} \to 1\). Thus, \(\lim_{x \to \infty} \frac{x}{[x]} = 1\) by the Squeeze Theorem.

7. \(f\) is continuous on \((-\infty, a)\) and \((a, \infty)\). To make \(f\) continuous on \(\mathbb{R}\), we must have continuity at \(a\). Thus,
\[
\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \Rightarrow \lim_{x \to a^+} x^2 = \lim_{x \to a^-} (x + 1) \Rightarrow a^2 = a + 1 \Rightarrow a^2 - a - 1 = 0 \Rightarrow
\]
by the quadratic formula, \(a = (1 \pm \sqrt{5})/2 \approx 1.618 \) or \(-0.618\).

9. \(\lim_{x \to a} f(x) = \lim_{x \to a} \left(\frac{1}{2} [f(x) + g(x)] + \frac{1}{2} [f(x) - g(x)]\right) = \frac{1}{2} \lim_{x \to a} [f(x) + g(x)] + \frac{1}{2} \lim_{x \to a} [f(x) - g(x)]
\]

= \(\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}\),

and \(\lim_{x \to a} g(x) = \lim_{x \to a} \left(\left\lfloor f(x) + g(x)\right\rfloor - f(x)\right) = \left\lfloor \lim_{x \to a} [f(x) + g(x)] - \lim_{x \to a} f(x)\right\rfloor = 2 - \frac{3}{2} = \frac{1}{2}.

So \(\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right] = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}\).

Another solution: Since \(\lim_{x \to a} [f(x) + g(x)]\) and \(\lim_{x \to a} [f(x) - g(x)]\) exist, we must have
\[
\lim_{x \to a} [f(x) + g(x)]^2 = \left(\lim_{x \to a} [f(x) + g(x)]\right)^2 \text{ and } \lim_{x \to a} [f(x) - g(x)]^2 = \left(\lim_{x \to a} [f(x) - g(x)]\right)^2,
\]
so
\[
\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} \frac{1}{2} [(f(x) + g(x))^2 - (f(x) - g(x))^2] \text{ [because all of the } f^2 \text{ and } g^2 \text{ cancel]}
\]

= \(\frac{1}{2} \left(\lim_{x \to a} [f(x) + g(x)]^2 - \lim_{x \to a} [f(x) - g(x)]^2\right) = \frac{1}{4} (2^2 - 1^2) = \frac{3}{4}\).

11. (a) Consider \(G(x) = T(x + 180^\circ) - T(x)\). Fix any number \(a\). If \(G(a) = 0\), we are done: Temperature at \(a = \) Temperature at \(a + 180^\circ\). If \(G(a) > 0\), then \(G(a + 180^\circ) = T(a + 360^\circ) - T(a + 180^\circ) = T(a) - T(a + 180^\circ) = -G(a) < 0\).

Also, \(G\) is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, \(G\) has a zero on the interval \([a, a + 180^\circ]\). If \(G(a) < 0\), then a similar argument applies.
(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

13. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \Rightarrow f(0) = 2f(0)$.

Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(0) + f(h) + 0^2h + 0h^2} {h} - f(0) = \lim_{h \to 0} \frac{f(h)} {h} = \lim_{x \to 0} \frac{f(x)} {x} = 1$

(c) $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) + f(h) + x^2h + xh^2} {h} - f(x) = \lim_{h \to 0} \frac{f(h) + x^2h + xh^2} {h}$

$= \lim_{h \to 0} \left[ \frac{f(h)} {h} + x^2 + xh \right] = 1 + x^2$